OVERSAMPLING OF STOCHASTIC PROCESSES

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In the theory of stochastic differential equations, it is commonly assumed that the forcing function is a Wiener process. Such a process has an infinite bandwidth in the frequency domain. In practice, however, all stochastic processes have a limited bandwidth.

A theory of band-limited linear stochastic processes is described that reflects this reality, and it is shown how the corresponding ARMA models can be estimated. By ignoring the limitation on the frequencies of the forcing function, in the process of fitting a conventional ARMA model, one is liable to derive estimates that are severely biased.

If the data are sampled too rapidly, then maximum frequency in the sampled data will be less than the Nyquist value. However, the underlying continuous function can be reconstituted by sinc function or Fourier interpolation; and it can be resampled at a lesser rate corresponding to the maximum frequency of the forcing function.

Then, there will be a direct correspondence between the parameters of the band-limited ARMA model and those of an equivalent continuous-time process; and the estimation biases can be avoided.

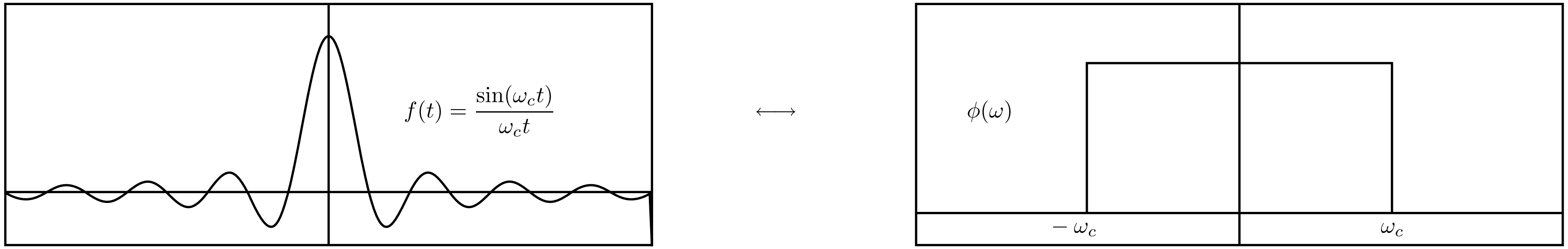
1. Time-Limited versus Band-Limited Processes

Stochastic processes in continuous time are usually modelled by filtered versions of Wiener processes which have infinite bandwidth. This seems inappropriate for modelling the slowly evolving trajectories of macroeconomic data. Therefore, we shall model these as processes that are limited in frequency.

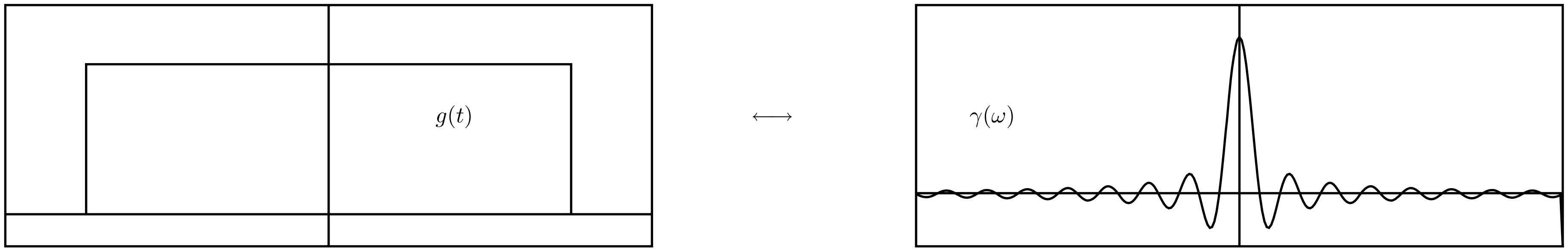
A function cannot be simultaneously limited in frequency and limited in time. One must choose either a band-limited function, which extends infinitely in time, or a time-limited function, which extents over an infinite range of frequencies.

A band-limited function is analytic. It possesses derivatives of all orders. Knowing its derivatives, allows one to find the turning points. It should also be possible to extrapolate the function indefinitely with perfect accuracy.

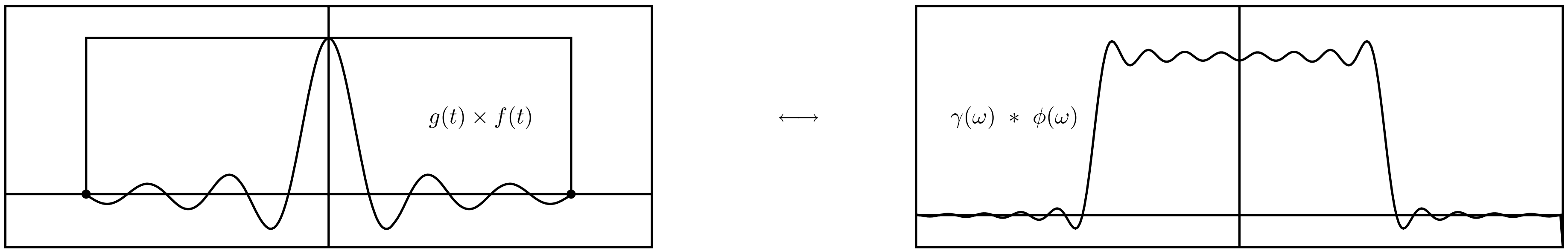
In a statistical context, the perfect predictability of band-limited functions may lead one to question their relevance. Eventually, I shall dispel the awkward implication of perfect predictability.



The Fourier Transform of a time-domain sinc function is a rectangle in the frequency domain.



The Fourier Transform of a time-domain rectangle is a sinc function in the frequency domain.



Truncating the time domain sinc function causes leakage in the frequency domain. The time-domain truncation corresponds to a convolution in the frequency domain.

2. Sampling in the Frequency Domain: Time Domain Wrapping

Sampling a function in the frequency domain has the effect of creating a periodic function in the time domain.

Consider the rectangle in the frequency domain of unit height supported on the interval $[-\pi, \pi]$. Its Fourier transform is the sinc function

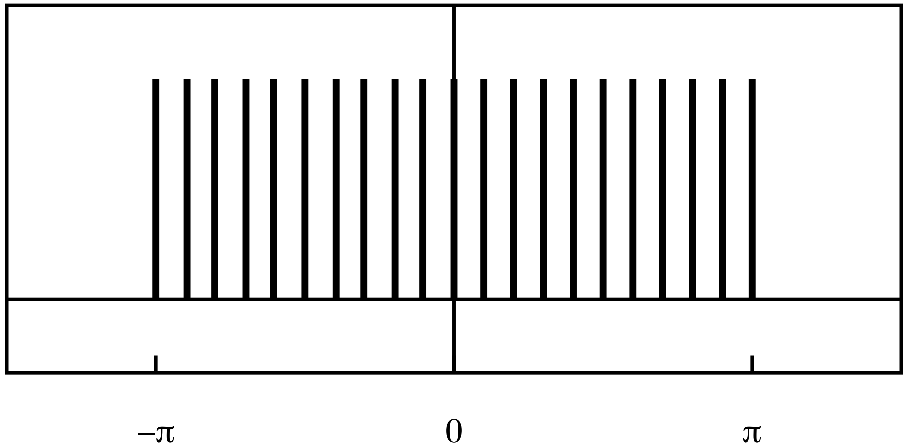
$$\phi(t) = \frac{\sin(\pi t)}{\pi t} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega(t)} d\omega.$$

Sampling the rectangle at $T = 2n$ points and applying the discrete Fourier transform to the resulting elements gives the Dirichlet kernel:

$$\phi^{\circ}(t) = \sin\frac{(\{[T-1]/2\}\omega_1 t)}{\sin(\omega_1 t/2)} = \sum_{j=-n}^{n-1} e^{i\omega_1 jt} = \sum_{j=0}^{T-1} e^{i\omega_1 jt},$$

where $\omega_1 = 2\pi/T$ is the fundamental frequency that relates to a function that completes a single cycle in T periods.

The Dirichlet kernel could be derived from the sinc function by wrapping it around a circle of radius T and adding the overlying ordinates.



The frequency-domain rectangle sampled at $M = 21$ points.

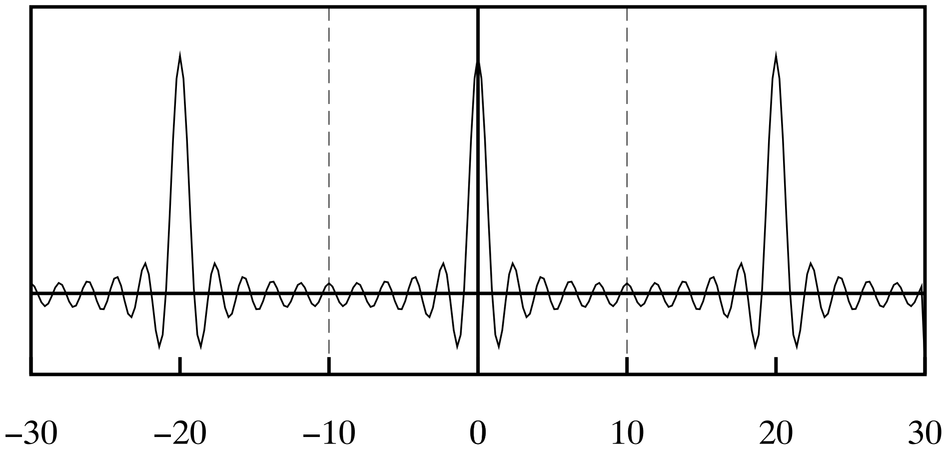


Figure 3. The Dirichlet function $\sin(\pi t)/\sin(\pi t/M)$ obtained from inverse Fourier transform of a frequency-domain rectangle sampled at $M = 21$ points

3. Sampling in the Time Domain: Frequency Domain Aliasing

A sequence sampled from a square-integrable continuous aperiodic function of time will have a periodic transform, with a period of 2π radians. Consider

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega \longleftrightarrow \xi(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}x(t)dt.$$

For a sample element of $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$, there is

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega \longleftrightarrow \xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-ik\omega}.$$

Therefore, at $x_t = x(t)$, there is

$$x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega.$$

The equality of the two integrals implies that

$$\xi_S(\omega) = \sum_{j=-\infty}^{\infty} \xi(\omega + 2j\pi).$$

If $\xi(\omega)$ is not band limited to $[-\pi, \pi]$, then $\xi_S(\omega) \neq \xi(\omega)$ and aliasing will occur.

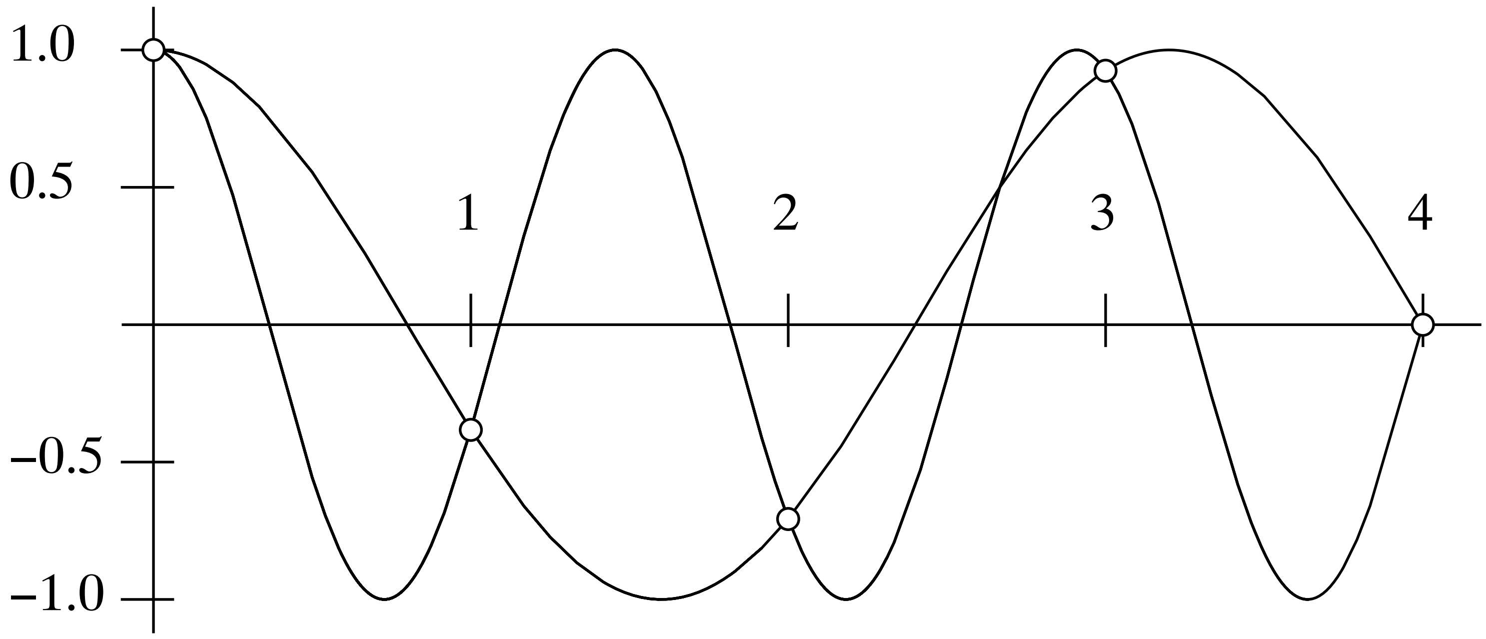


Figure 8. The values of the function $\cos\{(11/8)\pi t\}$ coincide with those of its alias $\cos\{(5/8)\pi t\}$ at the integer points $\{t = 0, \pm 1, \pm 2, \ldots\}$.

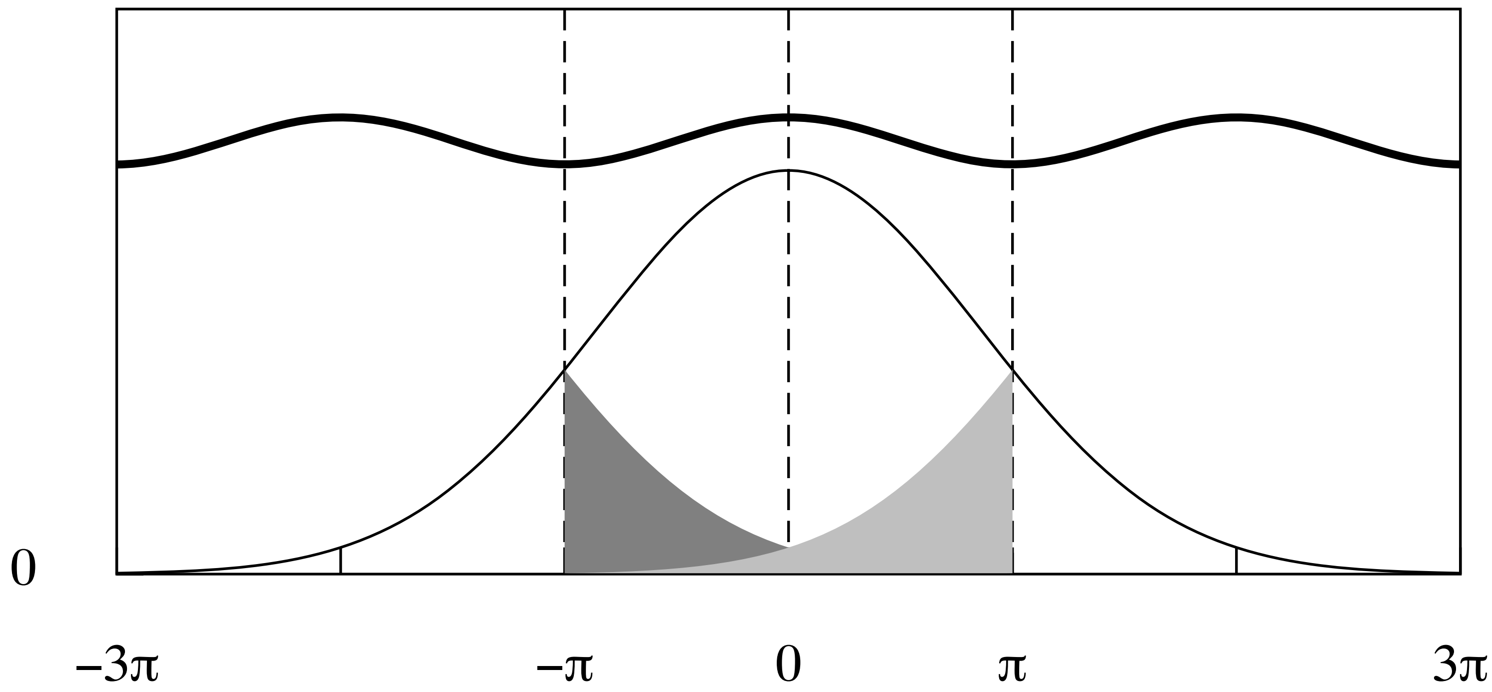


Figure 9. The figure illustrates the aliasing effect of regular sampling. The bell-shaped function supported on the interval $[-3\pi, 3\pi]$ is the spectrum of a continous-time process. The spectrum of the sampled process, represented by the heavy line, is a periodic function of period 2π.

The effect of sampling is to wrap the spectrum around a circle of radius 2π and to add the overlying parts. The same effect is obtained by folding the branches of the function, supported on $[-3\pi, -\pi]$ and $[\pi, 3\pi]$, onto the interval $[-\pi, \pi]$.

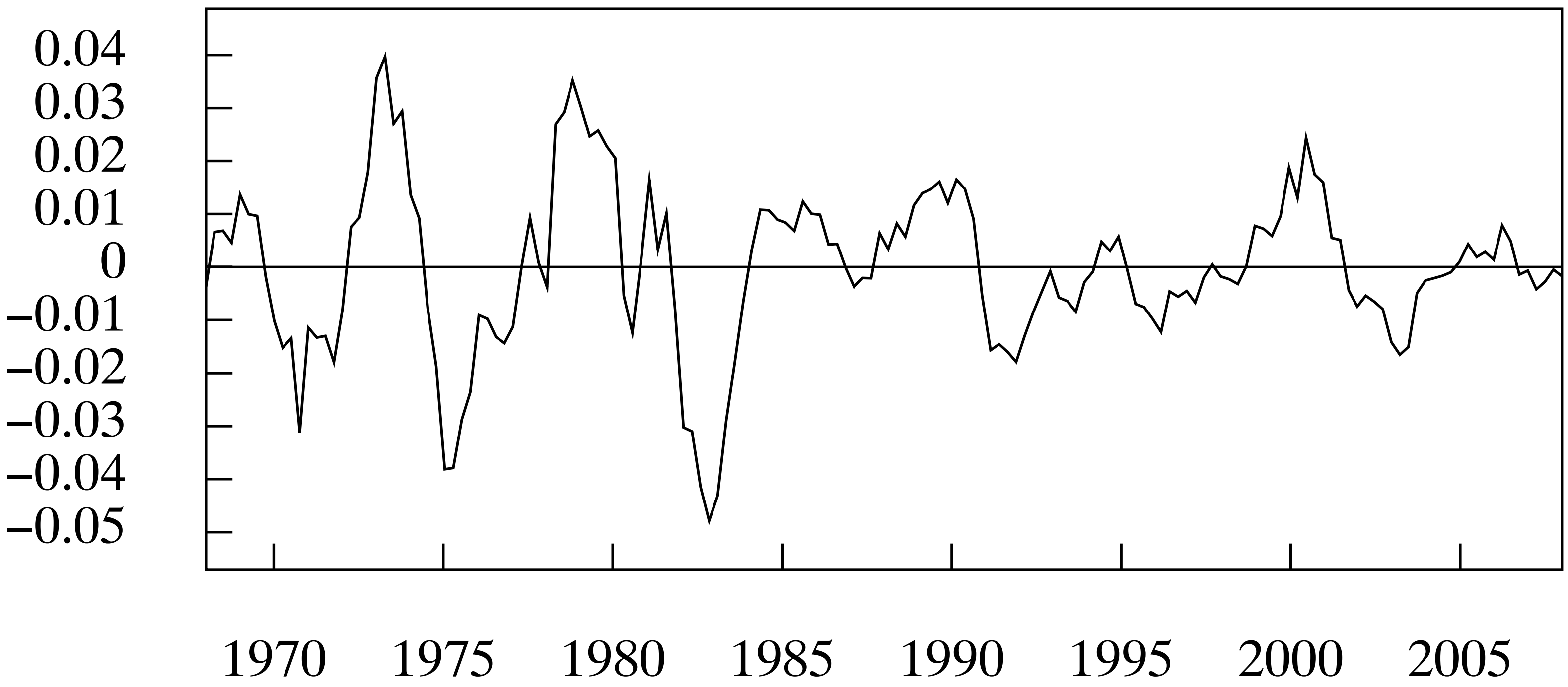


Figure. The deviations of the logarithmic quarterly index of real US GDP from an interpolated trend. The observations are from 1968 to 2007. The trend is determined by a Hodrick–Prescott (Leser) filter with a smoothing parameter of 1600.

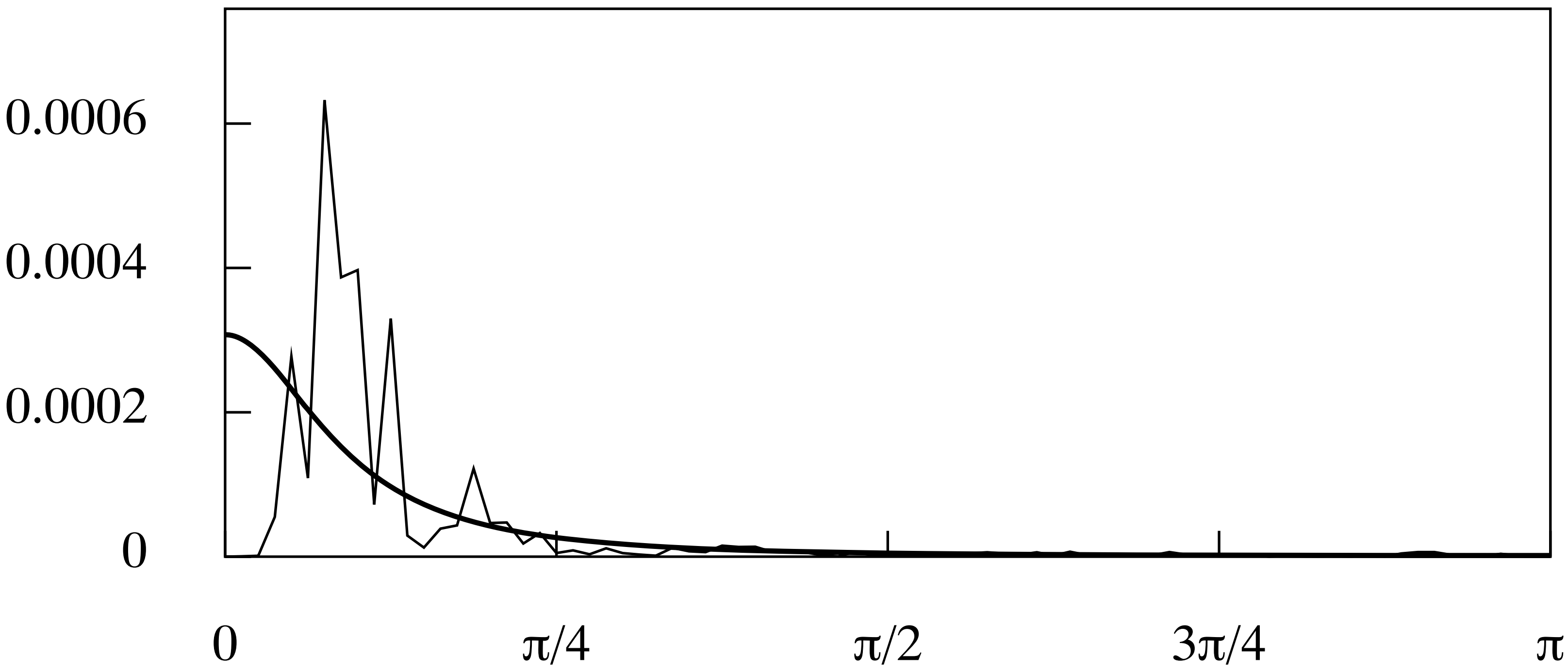


Figure. The periodogram of the data points of Figure 1 overlaid by the parametric spectral density function of an estimated regular AR(2) model.

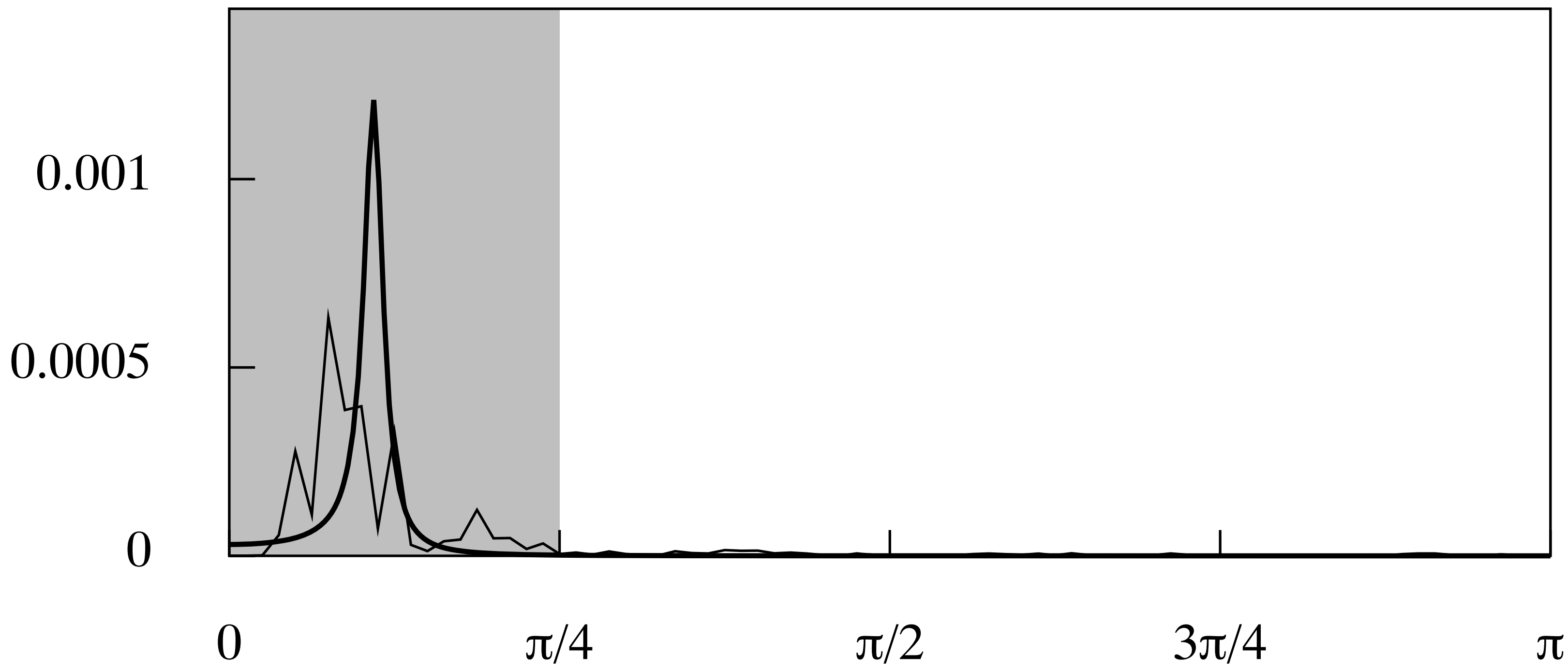


Figure. The periodogram of the data points of Figure 1 overlaid by the spectral density function of an AR(2) model estimated from band-limited data.

4. Sampling and Sinc-Function Interpolation

If $\xi(\omega) = \xi_S(\omega)$ is a continuous function band-limited to the interval $[-\pi, \pi]$, then it may be regarded as a periodic function of a period of 2π. Putting

$$\xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega} \quad \text{into} \quad x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega$$

gives

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega}\right\}e^{\mathrm{i}\omega t}d\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x_k\left\{\int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)}\right\}d\omega.$$

The integral on the RHS is evaluated as

$$\int_{-\pi}^{\pi} e^{i\omega(t-k)}d\omega = 2\frac{\sin\{\pi(t-k)\}}{t-k}.$$

Putting this into the RHS gives a weighted sum of sinc functions:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} x_k\phi_0(t-k), \quad \text{where} \quad \phi_0(t-k) = \frac{\sin\{\pi(t-k)\}}{\pi(t-k)}.$$

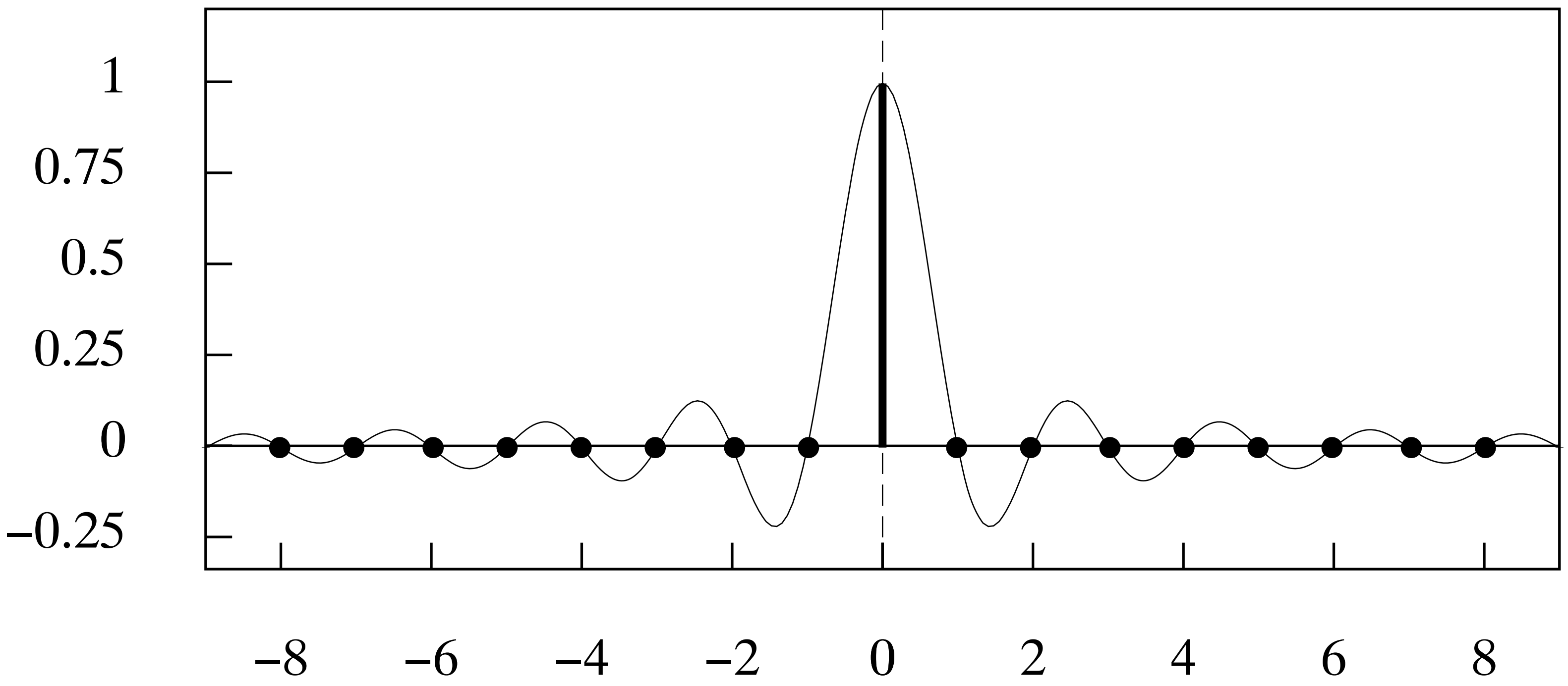


Figure. The sinc function wave-packet $\phi(t) = \sin(\pi t)/\pi t$ comprising frequencies in the interval $[0, \pi]$.

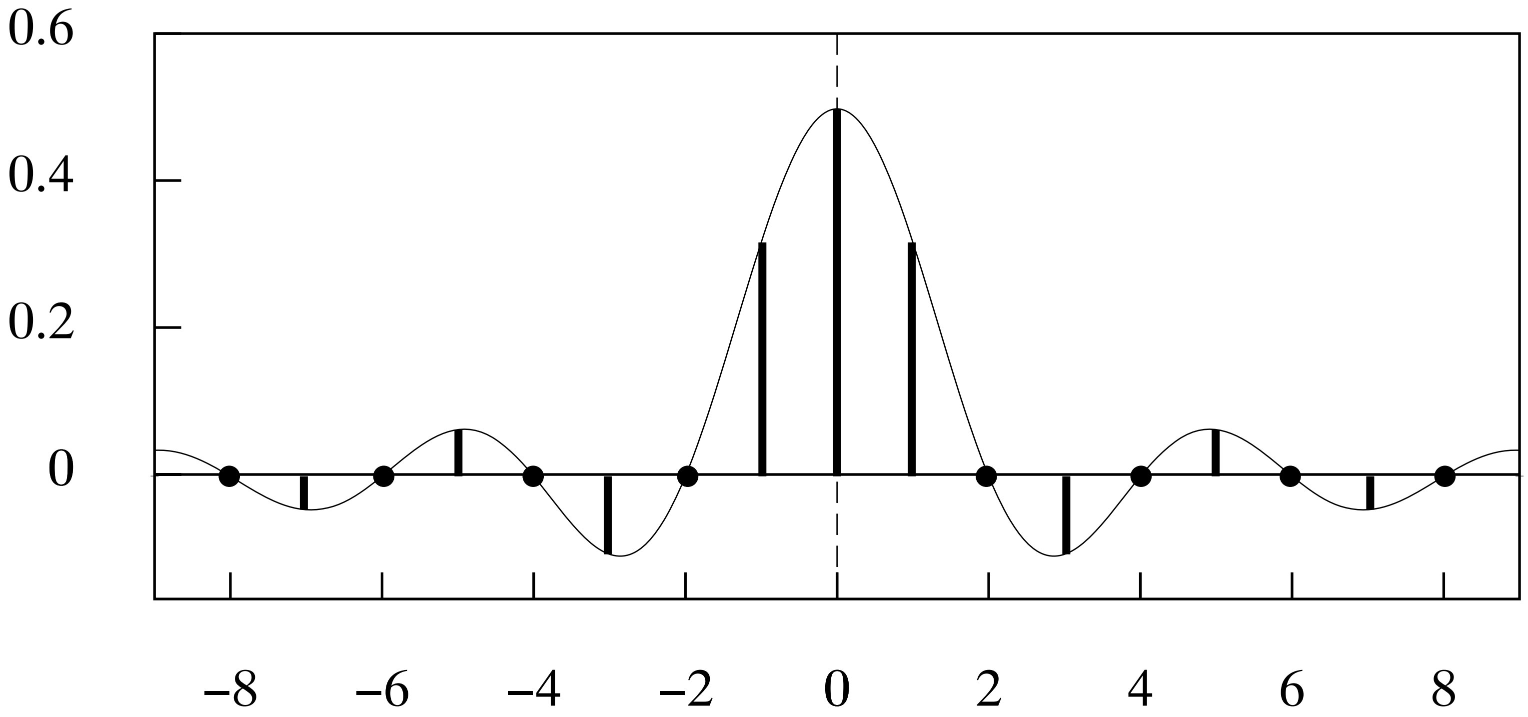


Figure 11. The sinc function wave-packet $\phi_1(t) = \sin(\pi t/2)/\pi t$ comprising frequencies in the interval $[0, \pi/2]$.

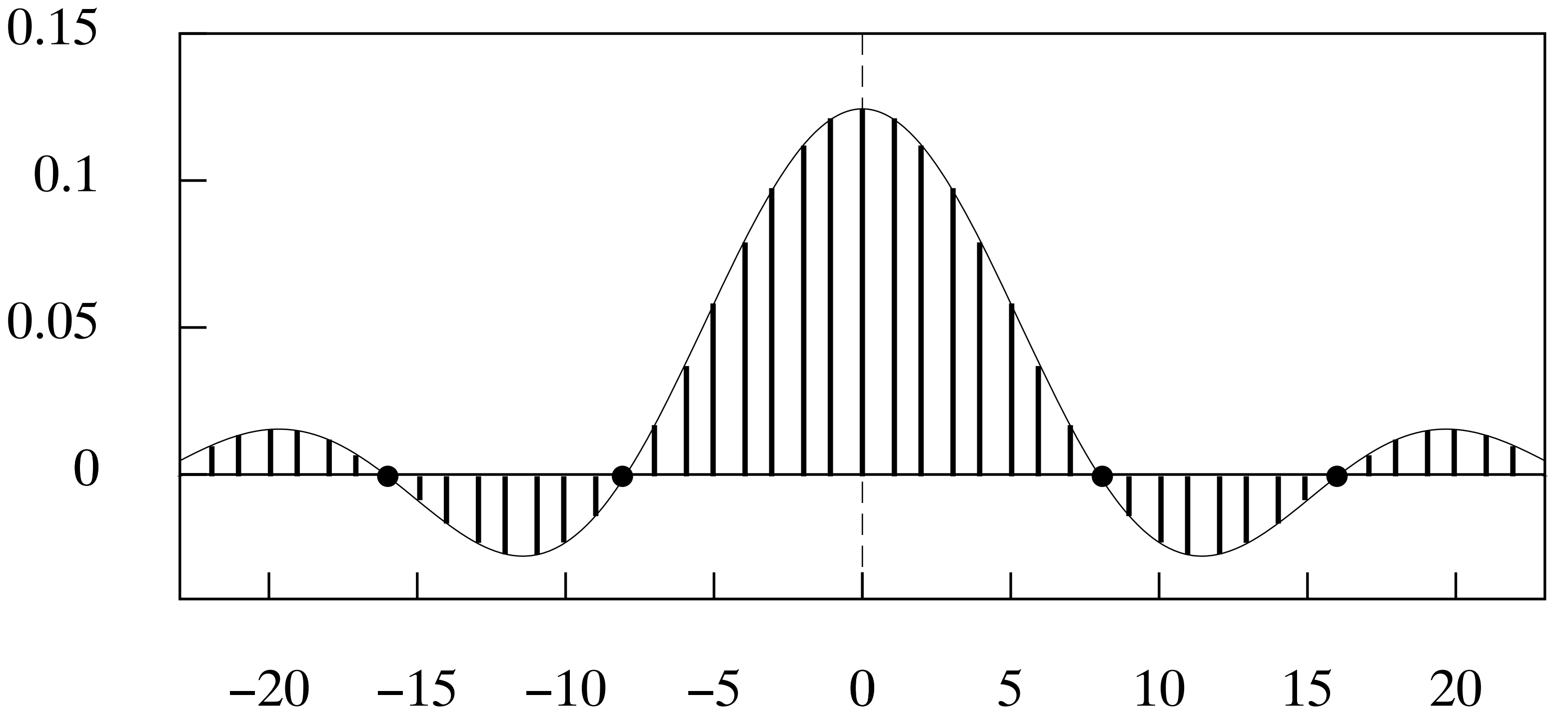


Figure 12. The sinc function wave-packet $\phi_3(t) = \sin(\pi t/8)/\pi t$ comprising frequencies in the interval $[0, \pi/8]$.

5. The Dirichlet Kernel

A function on a finite support can be regarded as a single cycle of a periodic function. The set of Dirichlet kernels at unit displacements provides the basis for the set of periodic functions limited in frequency to the Nyquist interval $[-\pi, \pi]$.

Let ξ_j^S be the jth ordinate from the discrete Fourier transform of $T = 2n$ points sampled from the function. If the function is band-limited to π radians, then there is

$$x(t) = \sum_{j=0}^{T-1} \xi_j^S e^{\mathrm{i}\omega_j t} \longleftrightarrow \xi_j^S = \frac{1}{T}\sum_{t=0}^{T-1} x_t e^{-\mathrm{i}\omega_j t}, \qquad \omega_j = \frac{2\pi j}{T}.$$

Putting the expression for the Fourier ordinates into the series expansion of the time-domain function and commuting the summation signs gives

$$x(t) = \sum_{j=0}^{T-1} \left\{ \frac{1}{T} \sum_{k=0}^{T-1} x_k e^{\mathrm{i}\omega_j k} \right\} e^{\mathrm{i}\omega_j t} = \frac{1}{T} \sum_{k=0}^{T-1} x_k \left\{ \sum_{j=0}^{T-1} e^{\mathrm{i}\omega_j (t-k)} \right\}.$$

The inner summation gives rise to the Dirichlet kernel:

$$\phi_n^{\circ}(t) = \sum_{t=0}^{T-1} e^{\mathrm{i}\omega_j t} = \frac{\sin([n - 1/2]\omega_1 t)}{\sin(\omega_1 t/2)}, \qquad \omega_1 = \frac{2\pi}{T}.$$

6. Fourier Interpolation

Thus the Fourier expansion can be expressed in terms of the Dirichlet kernel, which is a circularly wrapped sinc function:

$$x(t) = \frac{1}{T}\sum_{t=0}^{T-1} x_k \phi_n^\circ(t-k).$$

The functions $\{\phi^\circ(t-k); k = 0, 1, \ldots, T-1\}$ are appropriate for reconstituting a continuous periodic function $x(t)$ defined on the interval $[0, T)$ from its sampled ordinates $x_0, x_1, \ldots, x_{T-1}$. However, the periodic function can also be reconstituted by an ordinary Fourier interpolation

$$x(t) = \sum_{j=0}^{T-1} \xi_j e^{\mathrm{i}\omega_j t} = \sum_{j=0}^{[T/2]} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\},$$

where $[T/2]$ denotes the integral part of $T/2$ and where $\alpha_j = \xi_j - \xi_{-j}$ and $\beta_j = \mathrm{i}(\xi_j + \xi_{-j})$ are the coefficients from the regression of the data on the sampled ordinates of the cosine and sine functions at the various Fourier frequencies.

7. Band-Limited Stochastic Processes

If $\{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ are sampled ordinates of the ARMA process and if $\{\varepsilon_t; t = 0, \pm 1, \pm 2, \ldots\}$ are those of the white-noise forcing function, then the continuous functions are

$$y(t) = \sum_{k=-\infty}^{\infty} y_k \phi(t-k) \quad \text{and} \quad \varepsilon(t) = \sum_{k=-\infty}^{\infty} \varepsilon_k \phi(t-k),$$

where $t \in \mathcal{R}$ and $k \in \mathcal{Z}$ and where $\phi(t)$ is the sinc function The continuous-time ARMA process is

$$\sum_{j=0}^{p} \alpha_j y(t-j) = \sum_{j=0}^{q} \mu_j \varepsilon(t-j),$$

where $\alpha_0 = 1$. This has a moving-average representation in the form of

$$y(t) = \sum_{j=0}^{\infty} \psi_j \varepsilon(t-j),$$

where the coefficients are from the series expansion of the rational function $\mu(z)/\alpha(z) = \psi(z)$.

8. The Autocovariance Functions

The autocovariance function of the band-limited white-noise process is

$$\gamma_\varepsilon(\tau) = \sigma_\varepsilon^2 \phi(\tau) = \sigma_\varepsilon^2 \frac{\sin(\pi\tau)}{\pi\tau},$$

where $\tau \in \mathcal{R}$. The autocovariance function of the continuous ARMA process is given by

$$\gamma(\tau) = E\left\{\left[\sum_{i=0}^{\infty} \psi_i \varepsilon(t-\tau-i)\right]\left[\sum_{j=0}^{\infty} \psi_j \varepsilon(t-j)\right]\right\} = \sum_{i=-\infty}^{\infty} \gamma_i \phi(\tau - i),$$

where $\gamma_i = \sigma_\varepsilon^2 \sum_j \psi_j \psi_{j+i}$ is the ith autocovariance of the discrete-time process.

The autocovariance function of the continuous-time process is given by the inverse Fourier integral transform of the spectrum $f(\omega)$. Thus

$$\gamma(\tau) = \int_{-\pi}^{\pi} e^{\mathrm{i}\omega\tau} f(\omega) d\omega = \int_0^{\pi} 2\cos(\omega\tau) f(\omega) d\omega,$$

This must be approximated via a discrete cosine Fourier transform:

$$\gamma(\tau) \simeq \gamma_N^\circ(\tau) = \frac{2\pi}{N} \sum_{j=0}^{[N/2]} \cos(\omega_j \tau) f(\omega_j), \quad \omega_j = \frac{2\pi j}{N}.$$

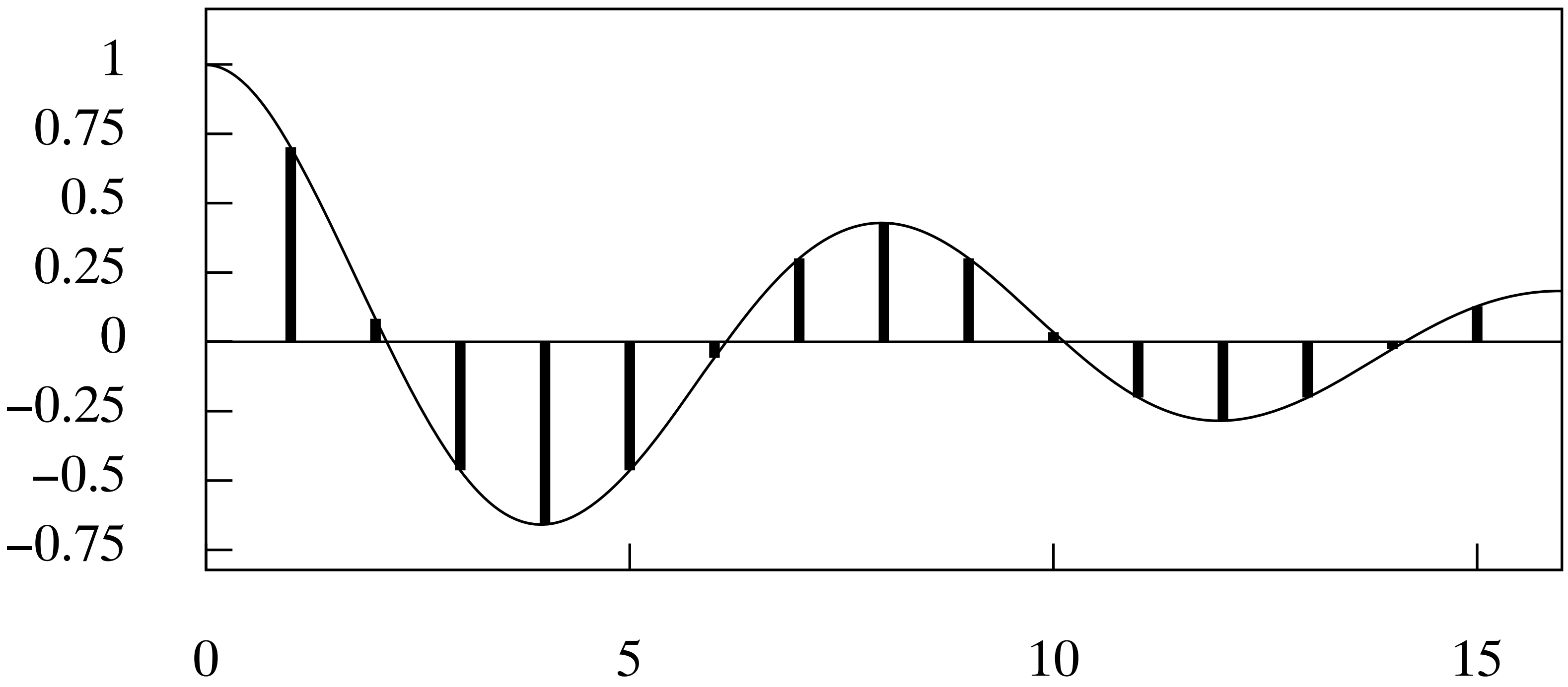


Figure. A continuous autocovariance function of an AR(2) process, obtained via the inverse Fourier transform of the spectral density function, together with the corresponding discrete-time autocovariances, calculated from the AR parameters.

9. Fitting Models to Oversampled Data

We propose to deal with problems of oversampled data by reducing the rate of sampling.

The first step is to reconstitute a continuous trajectory from the Fourier ordinates that lie within the frequency band in question.

The second step is to sample the continuous trajectory at the rate that is precisely attuned to the highest frequency that it contains. Then, an ARMA model can be fitted in the usual way to the resampled data.

Two computer programs are available at the address

`http://www.le.ac.uk/users/dsgp1/`

The program `BLIMDOS` generates pseudo-random band-limited data (i.e. oversampled data) from an ARMA model specified by the user. An ordinary ARMA model can be fitted to these data and the resulting biases can be assessed.

The program `OVERSAMPLE` samples the continuous autocovariance function of an ARMA model and it proceeds to infer the parameters of an ARMA model of specified orders from these sampled ordinates. In this way, it determines the probability limits of the misspecified estimators.

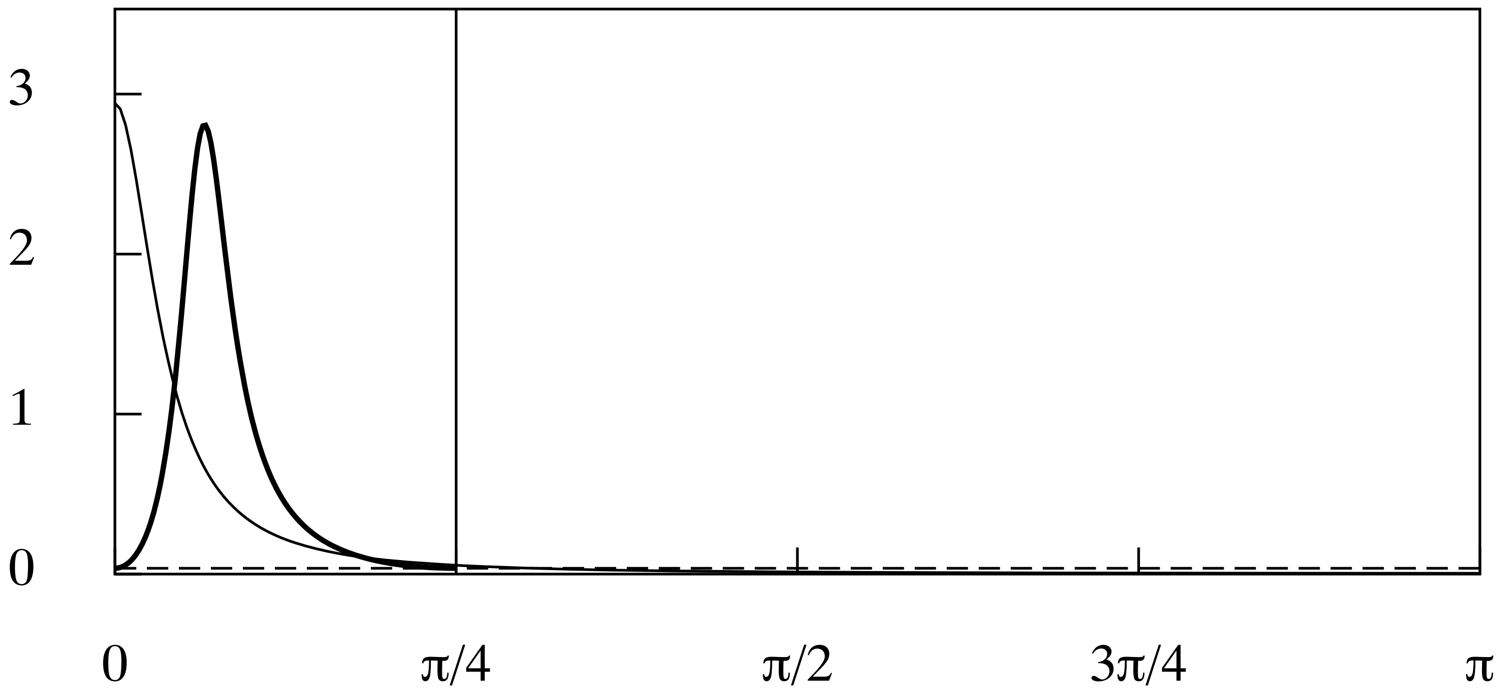


Figure. The parametric spectrum of the oversampled ARMA(2, 2) process, represented by a heavy line, supported on the spectrum of a white-noise contamination, together with the parametric spectrum of an AR(2) model fitted to the sampled autocovariances.

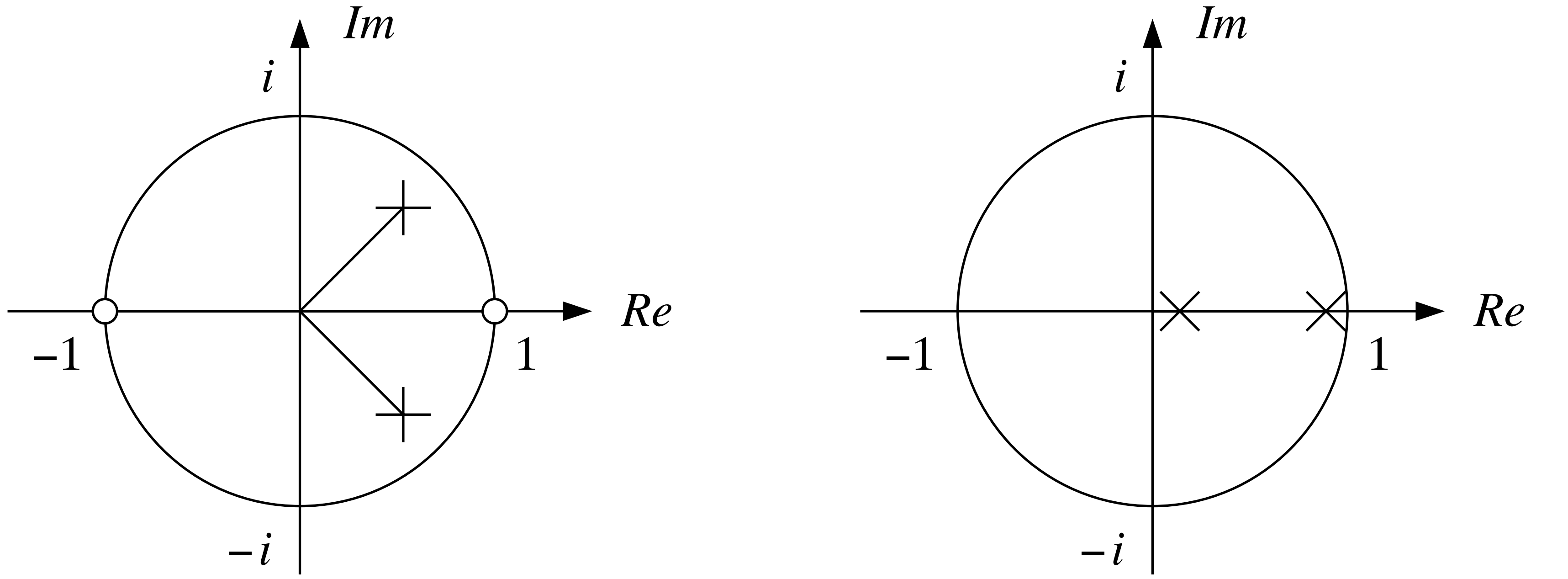


Figure. The pole-zero diagram of the ARMA(2, 2) model (left) and the diagram showing location of the poles of the AR(2) model estimated from band-limited data contaminated by white-noise errors of observation (right).

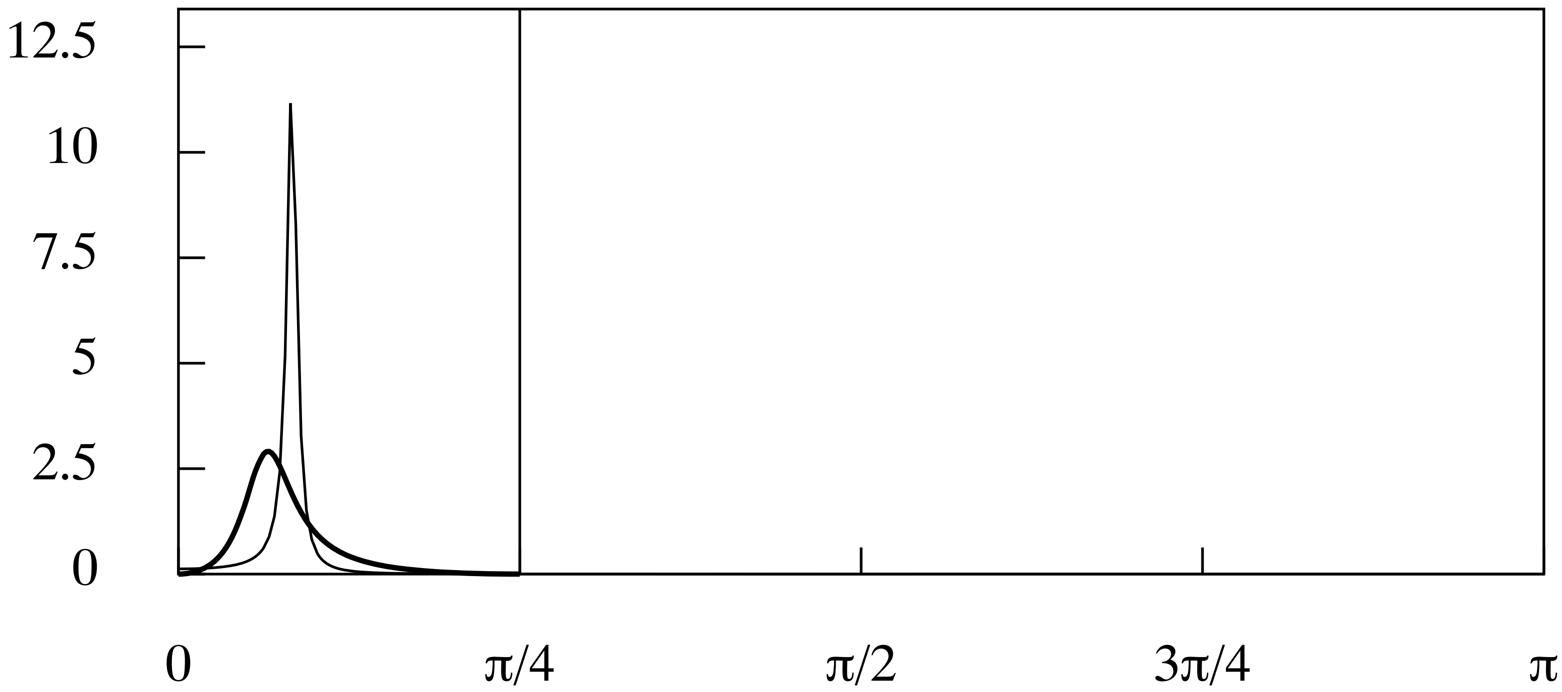


Figure. The parametric spectrum of an ARMA(2, 2) process, limited in frequency to π radians per period and oversampled at the rate of 4 observations per period, represented by a heavy line, together with the parametric spectrum of an AR(2) model fitted to the sampled autocovariances.

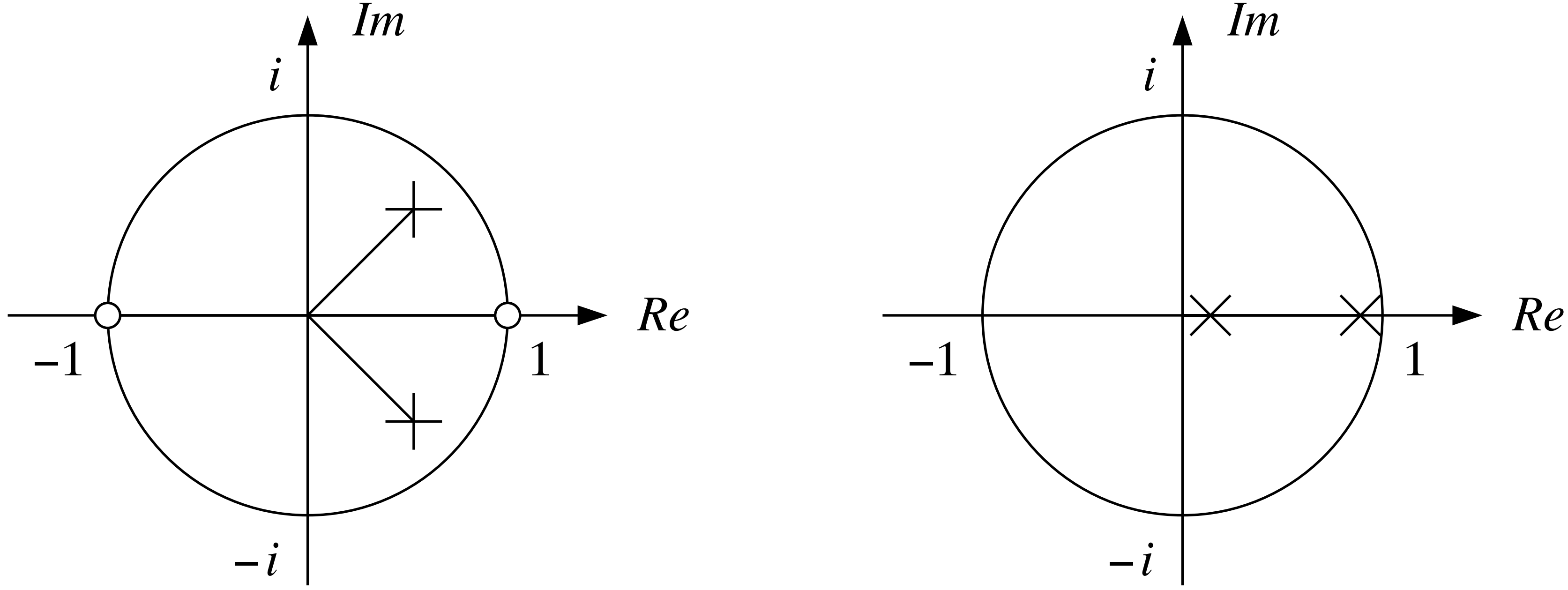


Figure. The pole-zero diagram of the ARMA(2, 2) model (left) and the diagram showing location of the poles of the AR(2) model estimated from band-limited data contaminated by white-noise errors of observation (right).

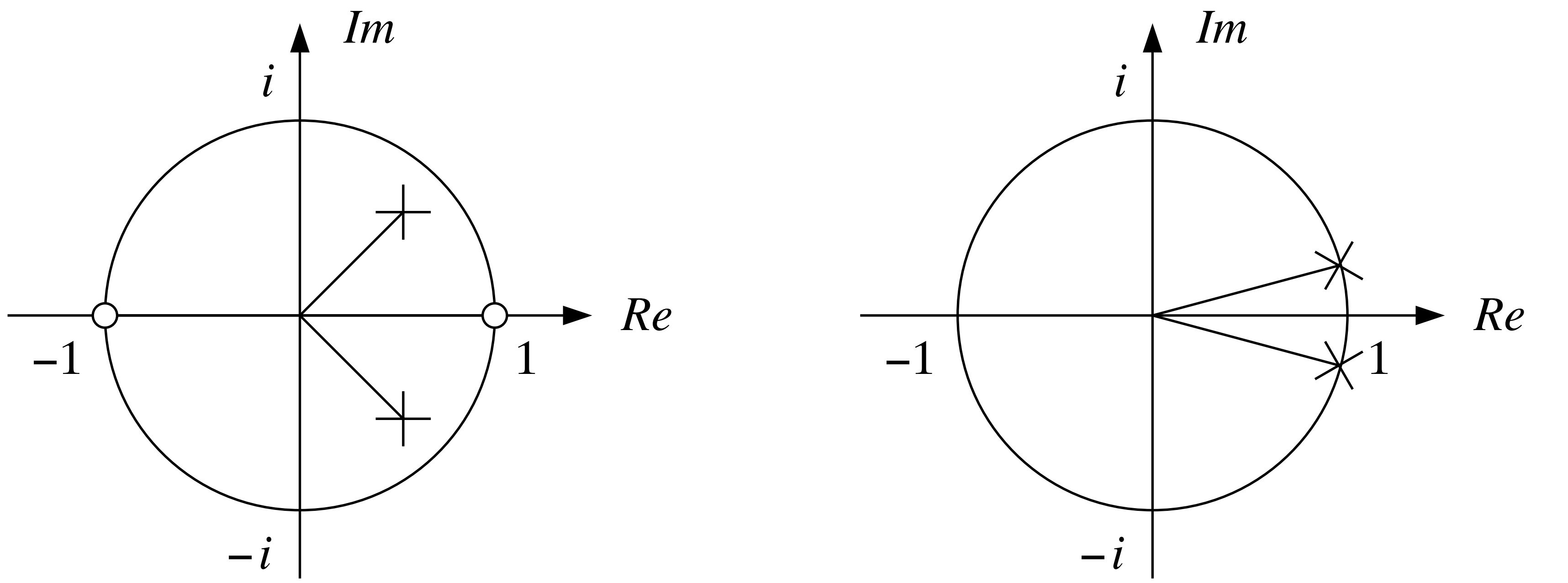


Figure. The pole-zero diagram of the ARMA(2, 2) model (left) and the diagram showing location of the poles of the AR(2) model estimated from band-limited data (right).

10. Explaining the Results

The results of these experiments can be explained by reference to the autocovariance function.

In the absense of noise contamination

When the rate of sampling is excessive, the autocovariances will be sampled at points that are too close to the origin, where the variance is to be found. Then, their values will decline at a diminished rate. The reduction in the rate of convergence is reflected in the modulus of the estimated complex roots, which understates the rate of damping.

When there is noise contamination

The variance of the white-noise errors of observation will be added to the variance of the underlying process. Nothing will be added to the adjacent sampled ordinates of autocovariance function. Therefore, the sampled autocovariances will decline at an enhanced rate.

If this rate of convergence exceeds the critical value, then there will be a transition from cyclical convergence to monotonic convergence, and the estimated autoregressive roots will be real-valued. This belies the cyclical nature of the true process.

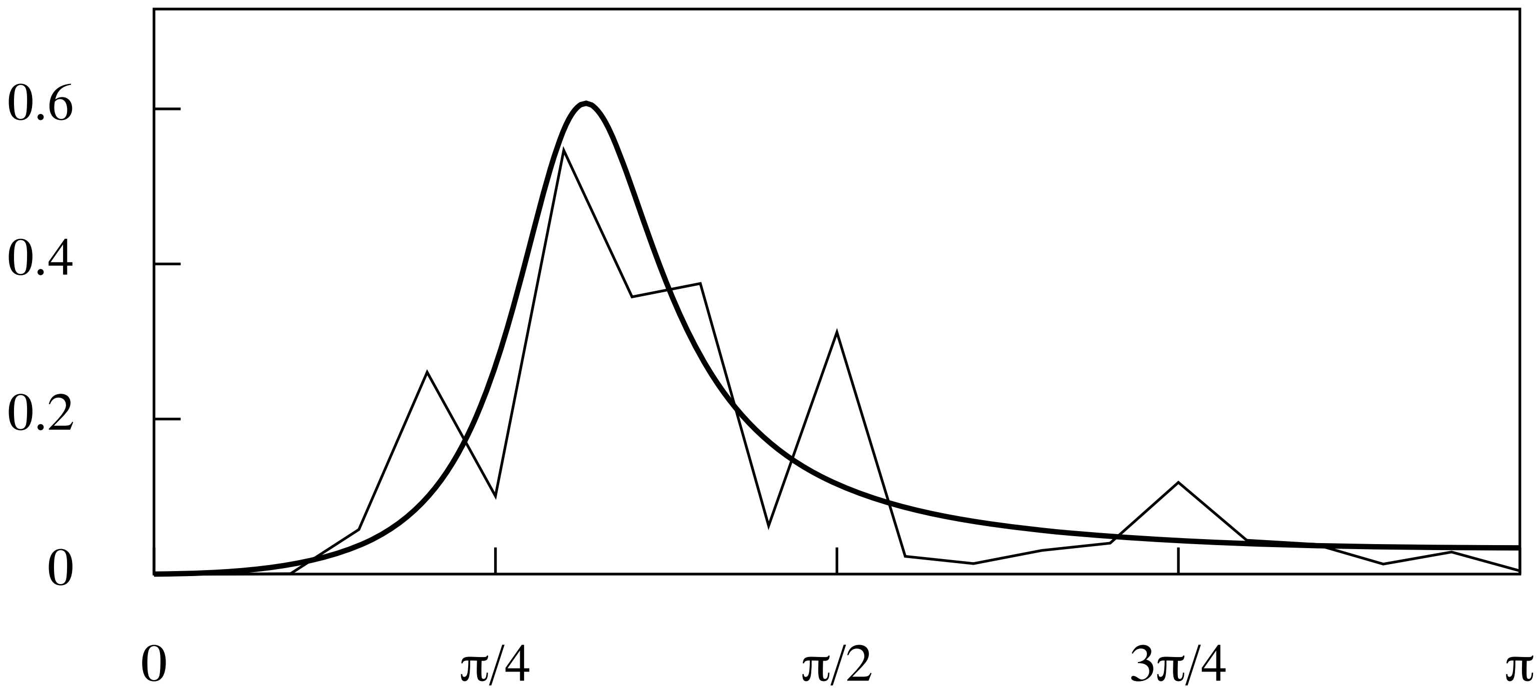


Figure. The periodogram of the data that have been filtered and subsampled at the rate of 1 observation in 4 overlaid by the parametric spectrum of an estimated ARMA(2, 1) model.

11. Characteristics of Band-Limited Processes

A function that is limited in frequency is also an analytic function. Such functions possess derivatives of all orders. Therefore the turning points of a frequency-limited business cycle trajectory can be located by finding where its derivatives are zero-valued.

A knowledge of a sufficient number of ordinates of an analytic function or of its derivatives should serve to specify the function completely. Therefore, it seems that band-limited stochastic processes should be perfectly predictable. The electrical engineers seem to be convinced of this.

To form a perfect prediction, one would need to have a denumerable infinity of sampled values, and there should be no errors of observation. The theoretical predictability of such a process depends on the infinite supports of the frequency-bounded sinc functions with the effect that every observation comprises traces of every sinc function.

The perfect predictability of band-limited processes is an analytic fantasy of the sort that Laplace derided in a famous passage, of which the intention is often misunderstood by Chaos theorists and others.

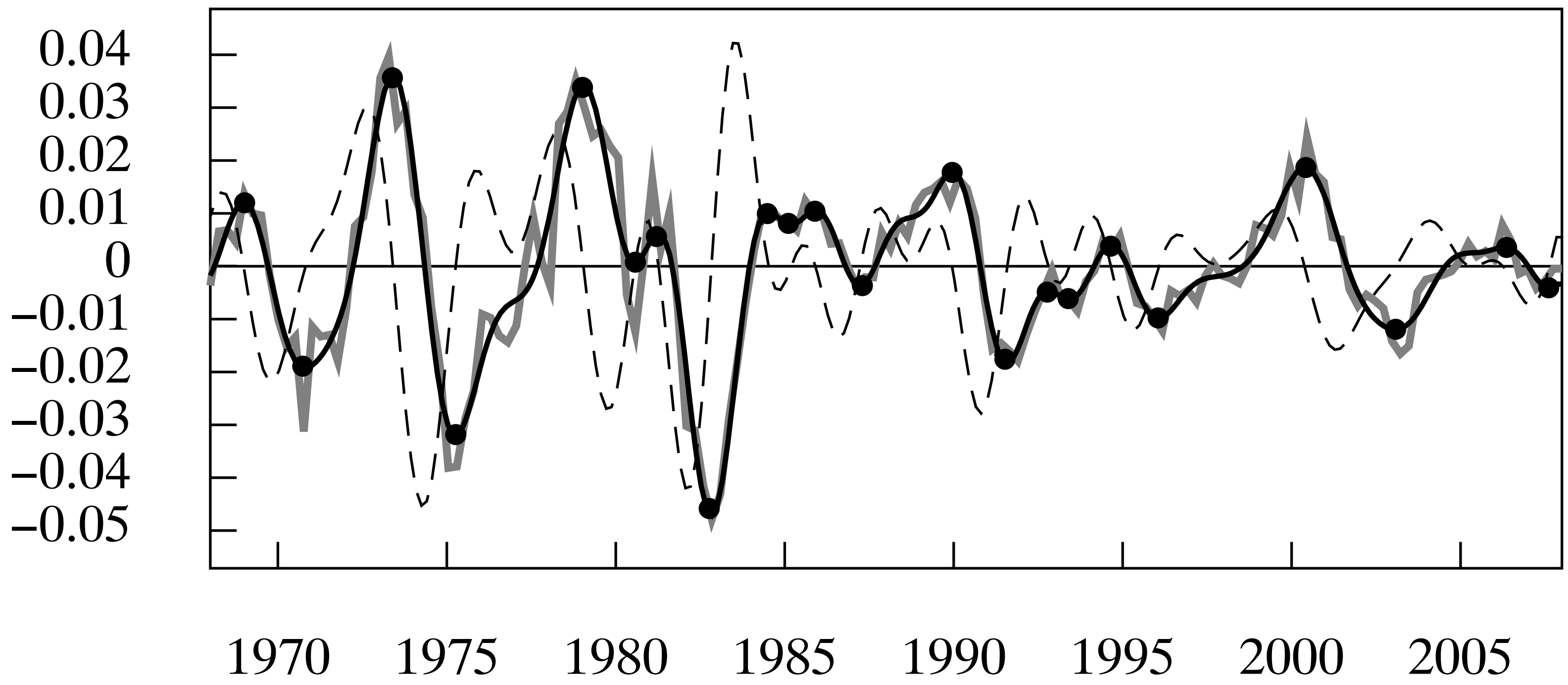


Figure. The turning points of the GDP index of Figure 1. The underlying trajectory is marked by the heavy line. Its turning points are marked by dots, which have been located by the zeros of the derivative function, which is represented by the discontinuous line. A plot of the original data is faintly visible beneath the curves.

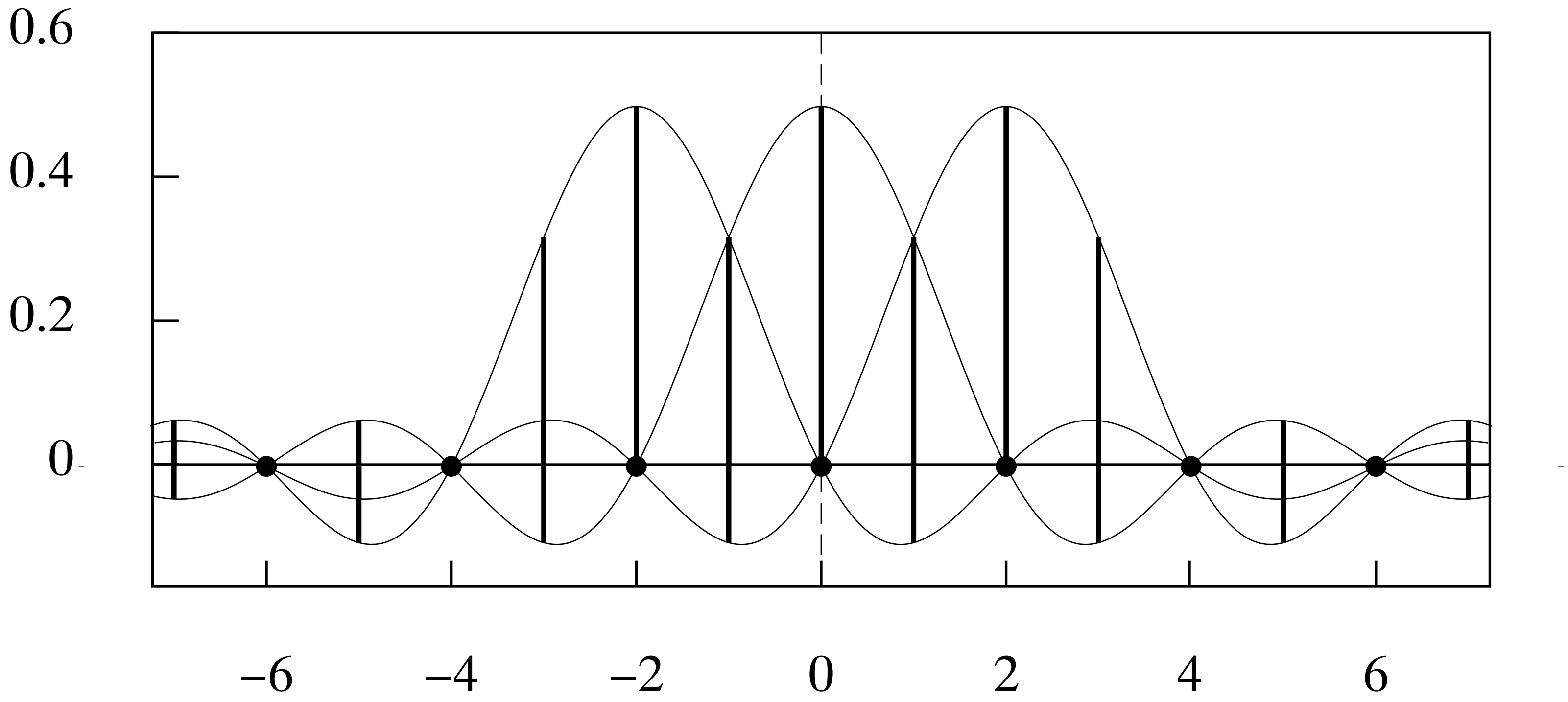


Figure 14. The wave packets $\phi_2(t)$ and $\phi_2(t-k)$ suffer no interference when $k \in \{\pm 2, \pm 4, \pm 6, \ldots\}$.

12. Lapalacian Determinism

Laplace accepted that the events of the universe must obey the laws of nature. Nevertheless, he proposed that a statistical approach is needed in describing these events, on account of our inability to comprehend more than a few of the innumerable factors and circumstances that affect each outcome:

> "We may regard the present state of the universe as the effect of its past and the cause of its future.
>
> An intellect which, at a certain moment, would know all forces that set nature in motion, and all positions of all items of which nature is composed, if it were also vast enough to submit these data to analysis, would embrace, in a single formula, the movements of the greatest bodies of the universe and those of the tiniest atom.
>
> For such an intellect, nothing would be uncertain and the future, as much as the past, would be present before its eyes."

Pierre-Simon Laplace, (1814), *Essai Philosophique sur les Probabilités.*