PROCESSES IN DISCRETE AND CONTINUOUS TIME

by D.S.G. POLLOCK
University of Leicester

I wish to discuss the relationship between discrete-time linear stochastic models and their continuous-time counterparts.

The original proponents of continuous-time econometrics were concerned with structural econometric models.

The identification of the equations requires *a priori* information, usually in the form of exclusion restrictions, which indicate that some of the system's variables are absent from some of its equations.

It was observed that, if such restrictions were valid for continuous-time models, then, on account of the feedback occurring in the time between successive observations, the restrictions would not be valid in discrete-time models.

Hence the requirement to model in continuous time.

My principal interest in the discrete–continuous relationship has a different motivation. I am interested in wavelets analysis and in digital signal processing, where the discrete–continuous correspondence in an essential feature.

A Misunderstanding of the Discrete–Continuous Relationship

There is a common misunderstanding that can be illustrated by quoting a passage from the Palgrave handbook on *Macroeconometrics and Time Series Analysis:*

> "A model built in continuous time can include discrete delays and discontinuities. But, only if all of the delays were discrete multiples of a single underlying time unit and synchronised across agents in the economy, would modelling with a discrete time unit be appropriate."

Whereas many econometric agents act in an instant under the impact of discrete events, their reactions are rarely precisely synchronised. When the reactions are taken in aggregate, the resulting indices are liable to display slowly evolving low-frequency trajectories, without discontinuities.

Moreover, the analysis overlooks the fundamental theorem that is at the heart of modern digital communications.

The Nyquist–Shannon sampling theorem indicates that, if at least two observations are taken in the time that it takes the signal component of highest frequency to complete a single cycle, then a continuous signal can be reconstructed perfectly from the sampled sequence.

The Meaning of the Sampling Theorem

An insight into the relationship between the discrete and the continuous began to emerge in the early years of cinematography.

A moving picture is created from a succession of images that capture instants in the trajectories of moving objects. In the early years of the cinema, the rate at which the images succeeded each other was too slow to produce a convincing depiction of smoothly continuous motion.

An allusion to early cinematography is conveyed by a famous painting by Marcel Duchamp titled *Nude Descending a Staircase,* which was exhibited in 1912 in Paris in the *Salon des Independents.*

When sampling is insufficiently rapid, the sample may be afflicted by the problem of aliasing, whereby high-frequency components of the signal are proxied by motions of lower frequency that fall within the resolution of the sample.

An example of aliasing, which is familiar to cinema goers of more than a certain age, concerns the image of a fleeing stage coach. The rapidly rotating wheels of the coach have a seemingly slow and sometimes retrograde motion.

The Shannon-Nyquist Sampling Theorem

If $\xi_S(\omega)$ is the Fourier transform of a continuous function $x(t)$ that is limited in frequency to the interval $[-\pi, \pi]$, then it may be regarded as a periodic function of period 2π. Putting

$$\xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega} \quad \text{into} \quad x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega$$

gives

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega}\right\} e^{\mathrm{i}\omega t}d\omega = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x_k \left\{\int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)}\right\} d\omega.$$

The integral on the RHS is evaluated as

$$\int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)}d\omega = 2\frac{\sin\{\pi(t-k)\}}{t-k}.$$

Putting this into the RHS gives a weighted sum of sinc functions:

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} x_k\varphi(t-k),$$

where

$$\varphi(t-k) = \frac{\sin\{\pi(t-k)\}}{\pi(t-k)}.$$

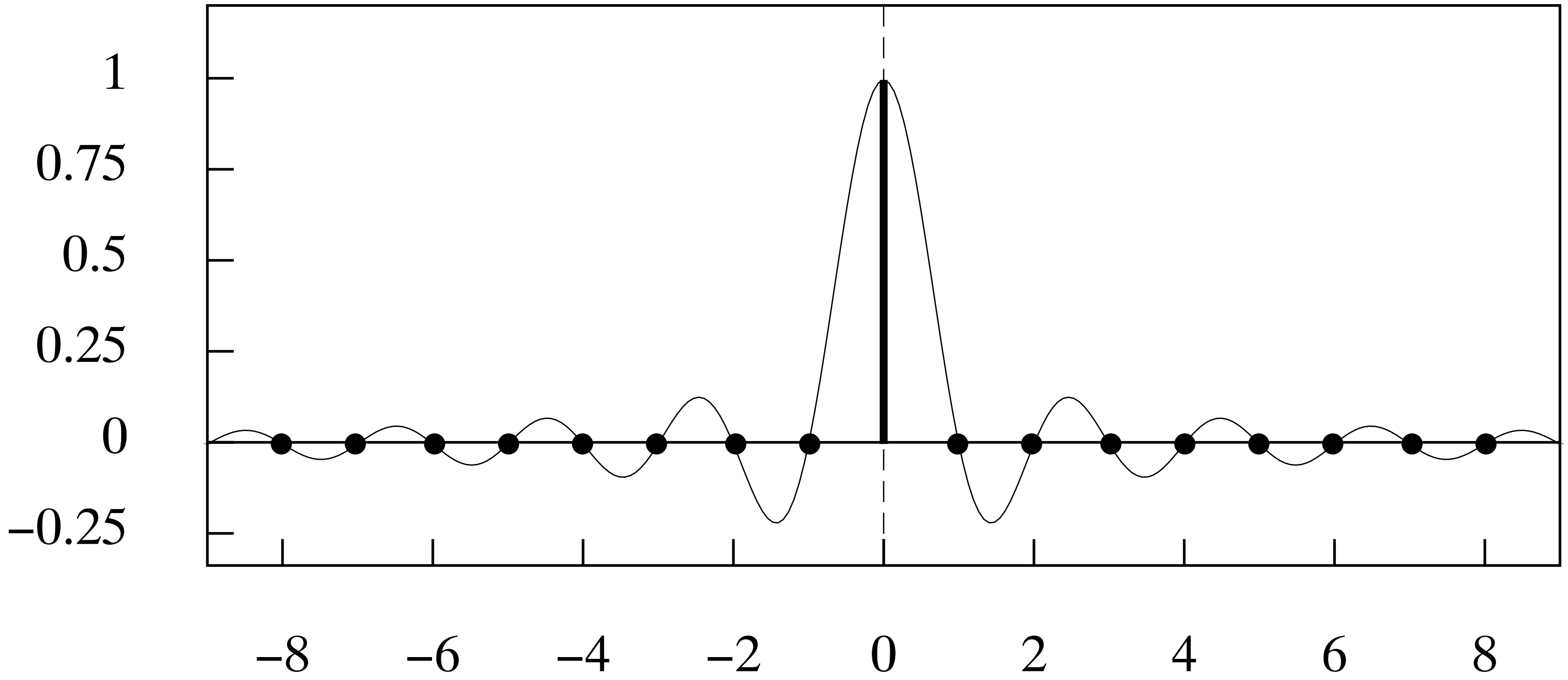

Figure 1. The sinc function wave-packet $\varphi(t) = \sin(\pi t)/\pi t$ comprising frequencies in the interval $[0, \pi]$.

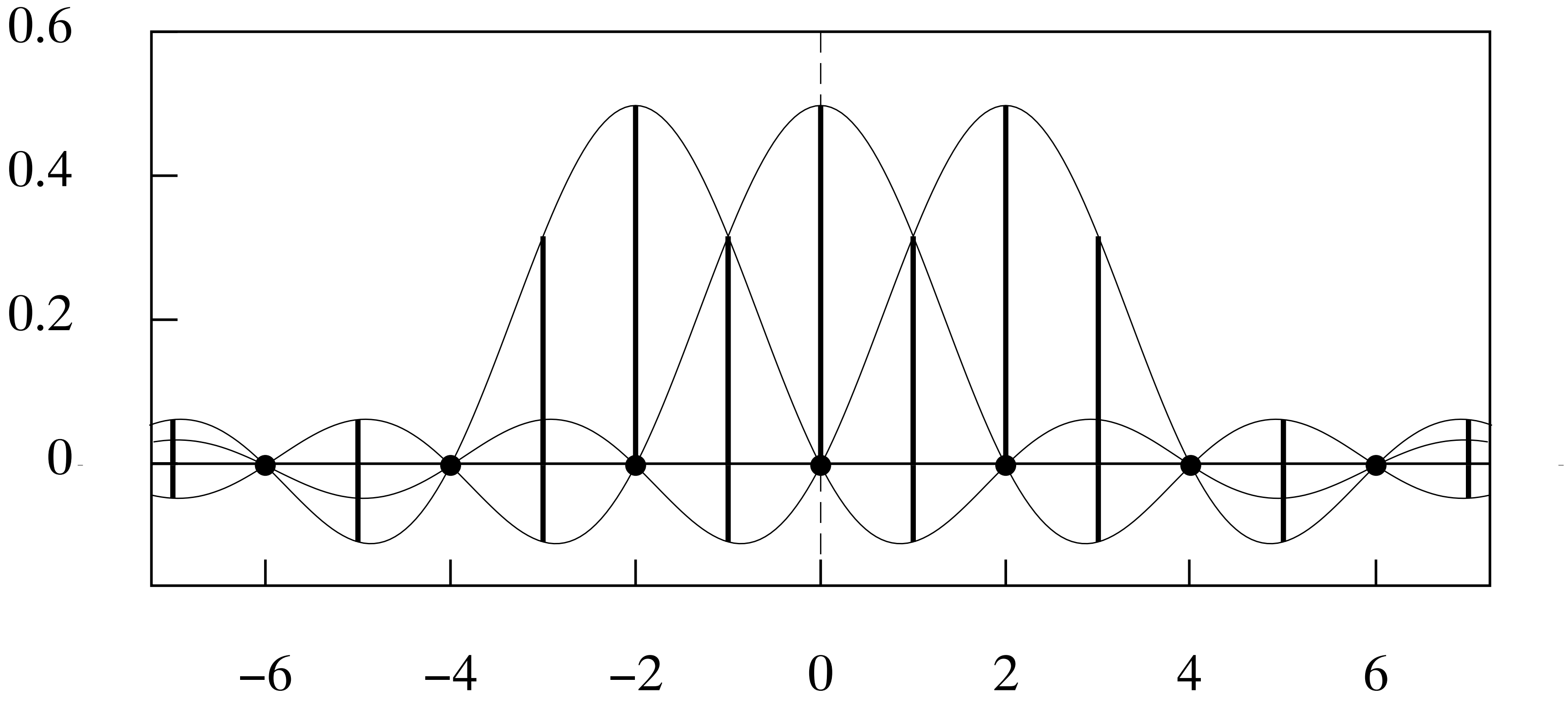

Figure 2. The wave packets $\varphi(t/2)$ and $\varphi(t/2 - k)$ suffer no interference when $k \in \{\pm 2, \pm 4, \pm 6, \ldots\}$.

The Wrapped Sinc Function and the Dirichlet Kernel

An apparent difficulty with this theorem lies in the fact that the sinc functions are supported on the entire real line.

Therefore, every sinc function that is indexed by the integers $\{k = 0, \pm 1 \pm 2, \ldots\}$, denoting their displacements, will be present at every point on the real line.

This means that the process of creating the continuous trajectory by adding the sinc functions of varying amplitudes and at successive displacements would entail an infinite sum.

However, the empirical data sequences, which are supported only on a finite set of T contiguous integer points, can be regarded as circular sequences. Therefore, the kernel functions that are to be applied to the sampled ordinates are circular or, equivalently, periodic sinc functions.

The sinc functions are wrapped around the circle of circumference T and their overlying ordinates are added to create so-called Dirichlet kernels. These kernels are applied to the circular data sequence in place of the sinc functions.

The set of Dirichlet kernels at unit displacements provides the basis for the set of periodic functions limited in frequency to the Nyquist interval $[-\pi, \pi]$.

The Wrapped Sinc Function and the Dirichlet Kernel

Let ξ_j^S be the jth ordinate from the discrete Fourier transform of $T = 2n$ points sampled from the function. If the function is frequency-limited to π radians, then there is

$$x(t) = \sum_{j=0}^{T-1} \xi_j^S e^{\mathrm{i}\omega_j t} \longleftrightarrow \xi_j^S = \frac{1}{T}\sum_{t=0}^{T-1} x_t e^{-\mathrm{i}\omega_j t}, \qquad \omega_j = \frac{2\pi j}{T}.$$

Putting the expression for the Fourier ordinates into the series expansion of the time-domain function and commuting the summation signs gives

$$x(t) = \sum_{j=0}^{T-1}\left\{\frac{1}{T}\sum_{k=0}^{T-1} x_k e^{\mathrm{i}\omega_j k}\right\} e^{\mathrm{i}\omega_j t} = \frac{1}{T}\sum_{k=0}^{T-1} x_k \left\{\sum_{j=0}^{T-1} e^{\mathrm{i}\omega_j (t-k)}\right\}.$$

The inner summation gives rise to the Dirichlet kernel:

$$\varphi_n^{\circ}(t) = \sum_{t=0}^{T-1} e^{\mathrm{i}\omega_j t} = \frac{\sin([n-1/2]\omega_1 t)}{\sin(\omega_1 t/2)}, \qquad \omega_1 = \frac{2\pi}{T}.$$

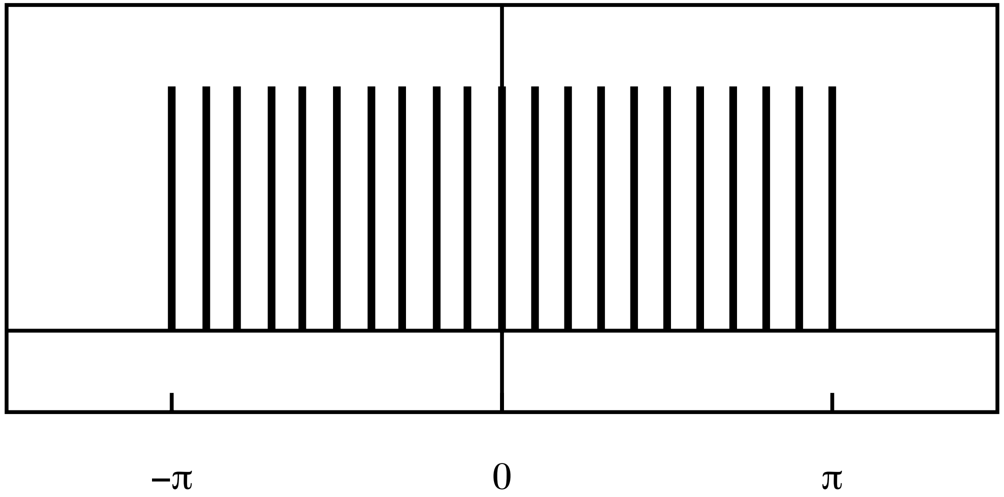

Figure 2. The frequency-domain rectangle sampled at $M = 21$ points.

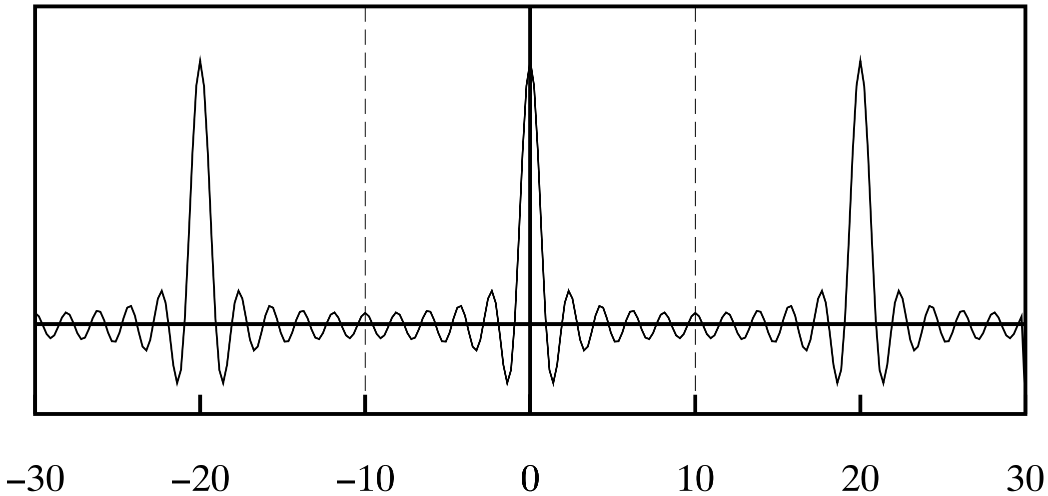

Figure 3. The Dirichlet function $\sin(\pi t)/\sin(\pi t/M)$ obtained from inverse Fourier transform of a frequency-domain rectangle sampled at $M = 21$ points.

Fourier Interpolation

What this shows is that the Fourier expansion can be expressed in terms of the Dirichlet kernel, which is a circularly wrapped sinc function:

$$x(t) = \frac{1}{T}\sum_{t=0}^{T-1} x_k \varphi_n^{\circ}(t-k).$$

The functions $\{\varphi^{\circ}(t-k); k = 0, 1, \ldots, T-1\}$ are appropriate for reconstituting a continuous periodic function $x(t)$ defined on the interval $[0, T)$ from its sampled ordinates $x_0, x_1, \ldots, x_{T-1}$.

However, the periodic function can also be reconstituted by an ordinary Fourier interpolation

$$x(t) = \sum_{j=0}^{T-1} \xi_j e^{\mathrm{i}\omega_j t} = \sum_{j=0}^{[T/2]} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\},$$

where $[T/2]$ denotes the integral part of $T/2$ and where $\alpha_j = \xi_j - \xi_{T-j}$ and $\beta_j = \mathrm{i}(\xi_j + \xi_{T-j})$ are the coefficients from the regression of the data on the sampled ordinates of the cosine and sine functions at the various Fourier frequencies.

Sampling, Wrapping and Aliasing

A sequence sampled from a square-integrable continuous aperiodic function of time will have a 2π-periodic transform. Consider

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega \longleftrightarrow \xi(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega t}x(t)dt.$$

For a sample element of $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$, there is

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega \longleftrightarrow \xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-ik\omega}.$$

Therefore, at $x_t = x(t)$, there is

$$x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}\omega t}\xi(\omega)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega.$$

The equality of the two integrals implies that

$$\xi_S(\omega) = \sum_{j=-\infty}^{\infty} \xi(\omega + 2j\pi).$$

If $\xi(\omega)$ is not limited in frequency to $[-\pi, \pi]$, then $\xi_S(\omega) \neq \xi(\omega)$ and aliasing will occur.

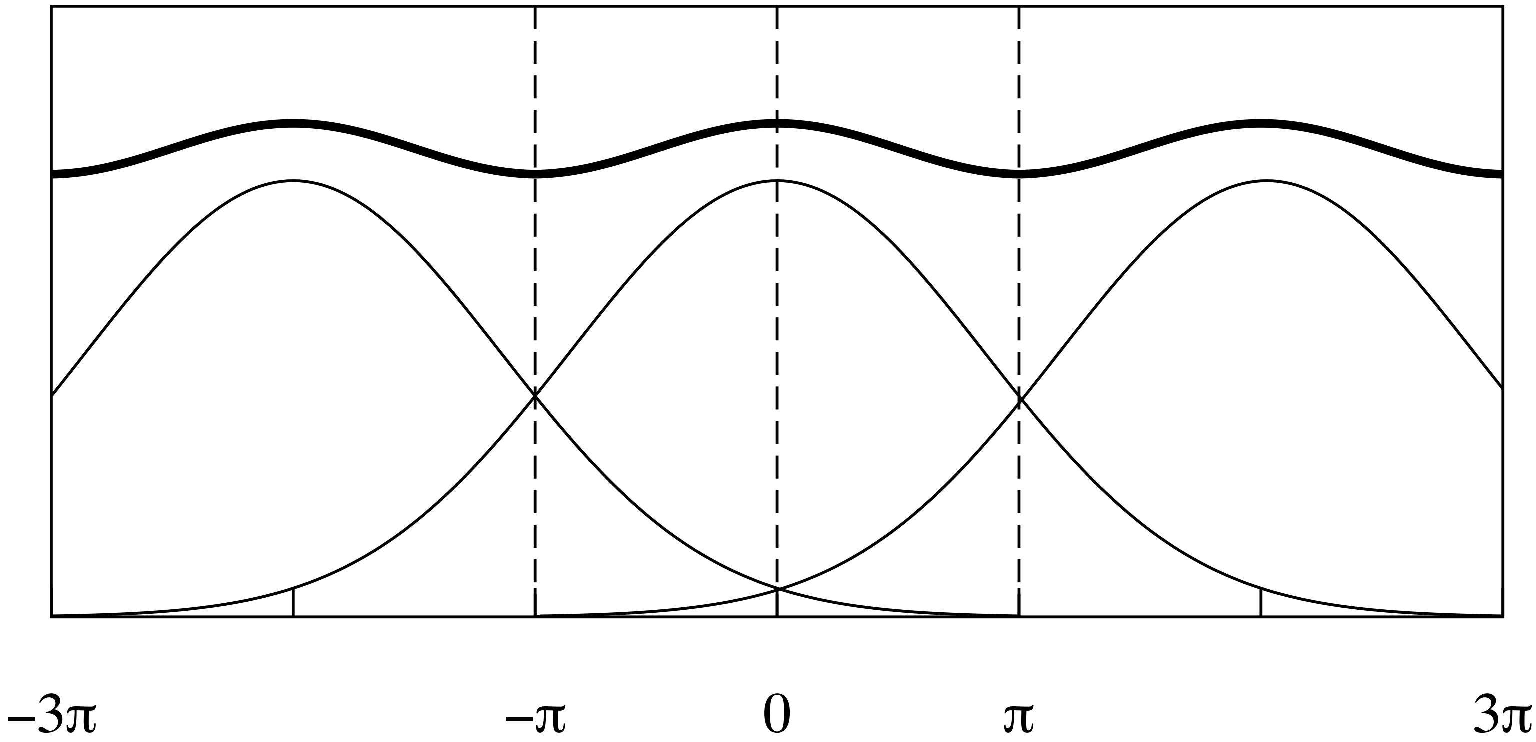

Figure 4. The figure illustrates the aliasing effect of regular sampling. The bell-shaped function supported on the interval $[-3\pi, 3\pi]$ is the spectrum of a continuous-time process. The spectrum of the sampled process, represented by the heavy line, is a periodic function of period 2π.

Discrete-time ARMA Models

Using a partial fraction decomposition, the discrete-time ARMA(p, q) model, with $p > q$, can be represented by

$$y(t) = \frac{\beta(L)}{\alpha(L)}\varepsilon(t) = \frac{\beta_0 + \beta_1 L + \cdots + \beta_q L^q}{1 + \alpha_1 L + \cdots + \alpha_p L^p}\varepsilon(t) = \left\{\sum_{i=1}^{p} \frac{d_i}{1 - \mu_i L}\right\}\varepsilon(t)$$

$$= \sum_{j=0}^{\infty}\{d_1\mu_1^j + d_2\mu_2^j + \cdots + d_p\mu_p^j\}\varepsilon(t-j),$$

where

$$\alpha(z) = \prod_{i=1}^{p}(1 - \mu_i z) \quad \text{and} \quad d_j = \frac{\beta(1/\mu_j)}{\prod_{i\neq j}^{p}(1 - \mu_i/\mu_j)}; j = 1, \ldots, p.$$

The general analytic expression for the autocovariance function of an ARMA process is

$$\gamma_d(\tau) = \sigma_\varepsilon^2 \sum_{i=1}^{p}\Big\{\sum_{j=1}^{p} \frac{d_i d_j}{1 - \mu_i\mu_j}\Big\}\mu_i^\tau.$$

Continuous-time ARMA Models

A continuous-time ARMA model supported on the Nyquist interval $[-\pi, \pi]$ can be derived from a discrete-time model by associating sinc functions to each of its ordinates.

If $\{y_k; k = 0, \pm 1, \pm 2, \ldots\}$ and $\{\varepsilon_k; k = 0, \pm 1, \pm 2, \ldots\}$ are the ordinates of the ARMA process and its forcing function, then the continuous-time functions are

$$y(t) = \sum_{k=-\infty}^{\infty} y_k \varphi(t-k) \quad \text{and} \quad \varepsilon(t) = \sum_{k=-\infty}^{\infty} \varepsilon_k \varphi(t-k),$$

respectively, where $t \in \mathcal{R}$ and $k \in \mathcal{Z}$ and where $\varphi(t)$ is the sinc function kernel.

The equation of the continuous-time ARMA process is

$$\sum_{j=0}^{p} \alpha_j y(t-j) = \sum_{j=0}^{q} \beta_j \varepsilon(t-j);, \quad \alpha_0 = 1.$$

This has a moving-average representation in the form of

$$y(t) = \sum_{j=0}^{\infty} \psi_j \varepsilon(t-j),$$

where the coefficients are from the series expansion of the rational function $\beta(z)/\alpha(z) = \psi(z)$.

Autocovariance Function of a Continuous-time ARMA Model

If $\varepsilon_t = \varepsilon(t)$ and $\varepsilon_s = \varepsilon(s)$, with $t, s \in \mathcal{R}$, are from the continuous frequency-limited white-noise forcing function, then their covariance is

$$C(\varepsilon_t, \varepsilon_s) = \sigma_\varepsilon^2 \varphi(t-s) = \gamma_\varepsilon(\tau), \quad \tau = t - s,$$

where σ_ε^2 is the variance parameter and $\varphi(\tau)$ is the sinc function. The result is understood by recognising that $\varepsilon_s = \varphi(\tau)\varepsilon_t + \eta$, where η is uncorrelated with ε_t.

If $y(t) = \sum_i \psi_i \varepsilon(t-i)$, and $y(s) = \sum_j \psi_j \varepsilon(s-j)$, with $t, s \in \mathcal{R}$ and $i, j \in \mathcal{Z}$, then the autocovariance function of $y_t = y(t)$ and $y_s = y(s)$ is

$$\begin{aligned} C\left\{ \sum_i \psi_i \varepsilon_{t-i}, \sum_j \psi_j \varepsilon_{s-j} \right\} &= \sum_i \sum_k \psi_i \psi_{i+k} C(\varepsilon_t, \varepsilon_{s-k}); \quad k = j - i \\ &= \sum_k \gamma_k \varphi(\tau - k) = \gamma(\tau); \quad \tau = t - s, \end{aligned}$$

where $\gamma_k = \sigma_\varepsilon^2 \sum_j \psi_j \psi_{j+j}$ is the kth autocovariance of the discrete-time process. Thus, the continuous-time autocovariance function can be obtained from the discrete-time function by sinc-function interpolation.

The Autocovariance Function and the Spectral Density Function

The spectrum $f(\omega)$ of the discrete-time process is a periodic function that is the Fourier transform of the sequence of autocovariances. It is generated by

$$\gamma(z) = \frac{\beta(z)\beta(z^{-1})}{\alpha(z)\alpha(z^{-1})}; \quad z = e^{i\omega},$$

as z travels around the perimeter of the unit circle in the complex plane.

The spectrum of the continuous-time ARMA process is limited in frequency to the Nyquist interval $[-\pi, \pi]$, where it takes the same values as that of the discrete-time process.

The autocovariance function of the continuous-time process is given by the inverse Fourier integral transform of this spectrum:

$$\gamma(\tau) = \int_{-\pi}^{\pi} e^{i\omega\tau} f(\omega) d\omega = \int_{0}^{\pi} 2\cos(\omega\tau) f(\omega) d\omega,$$

It is obtained more easily by allowing τ to vary continuously in the formula

$$\gamma(\tau) = \sigma_\varepsilon^2 \sum_{i=1}^{p} \Big\{ \sum_{j=1}^{p} \frac{d_i d_j}{1 - \mu_i \mu_j} \Big\} \mu_i^\tau .$$

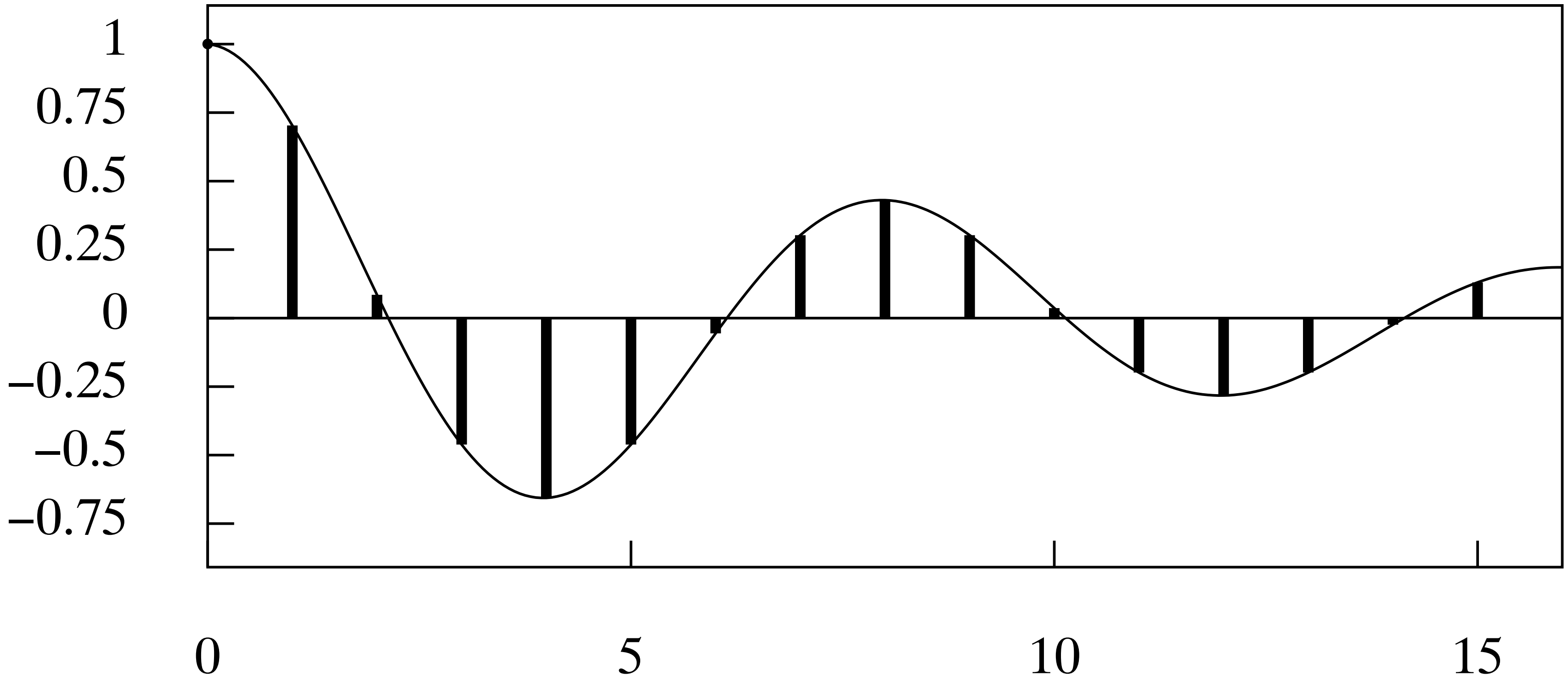

Figure 5. The autocovariance function of the an AR(2) process $(1 - 1.273L + 0.81L^2)y(t) = \varepsilon(t)$, rendered in both discrete and continuous forms.

Linear Stochastic Differential Equations

The Linear Stochastic Differential Equation (LSDE) that is commonly regarded as the continuous-time ARMA analogue is driven by the increments of a Wiener process. Such a process is unbounded in frequency.

The continuous-time LSDE(p, q) model is represented by

$$\begin{aligned} y(t) &= \frac{\theta(D)}{\phi(D)}\zeta(t) = \frac{\theta_0 D^q + \theta_1 D^{q-1} + \cdots + \theta_q}{\phi_0 D^p + \phi_1 D^p + \cdots + \phi_p}\zeta(t) = \left\{\sum_{i=1}^{p} \frac{c_i}{D - \kappa_i}\right\}\zeta(t) \\ &= \int_0^\infty \{c_1 e^{\kappa_1 t} + c_2 e^{\kappa_2 t} + \cdots + c_p e^{\kappa_p t}\}\zeta(t-\tau)d\tau, \end{aligned}$$

where

$$\phi(s) = \prod_{i=1}^{p}(s - \kappa_i) \quad \text{and} \quad c_j = \frac{\phi(\kappa_j)}{\prod_{i\neq j}(\kappa_j - \kappa_i)}.$$

Based on the transfer function $\psi(t) = \sum c_i e^{\kappa_i t}$, the autocovariance function is

$$\begin{aligned} \gamma_c(\tau) = \sigma_\zeta^2 \int_0^\infty \psi(t)\psi(t+\tau)dt &= \sigma_\zeta^2 \sum_{i=1}^{p}\sum_{j=1}^{p}\left\{c_i c_j \int_0^\infty e^{(\kappa_i+\kappa_j)t+\kappa_i\tau}dt\right\} \\ &= \sigma_\zeta^2 \sum_{i=1}^{p}\left\{\sum_{j=1}^{p} c_i c_j \frac{-e^{\kappa_i\tau}}{\kappa_i + \kappa_j}\right\}. \end{aligned}$$

Autocovariances of an LSDE(2, 1) Driven by Increments of a Wiener Process

The autocovariance function, which is real-valued, may be computed either in complex arithmetic or in real terms.

The LDSE(2, 1) with conjugate complex poles κ and κ^* will serve as an illustration. The products of the partial-fraction coefficients c and c^*, which are also complex conjugates, are tabulated as follows:

	c	c^*
c	cc	cc^*
c^*	c^*c	c^*c^*

When these coefficients are associated with the denominators, there is

$$\gamma_c(\tau) = -\sigma_\zeta^2 \left\{ \frac{cc}{2\kappa} e^{\kappa\tau} + \frac{c^*c^*}{2\kappa^*} e^{\kappa^*\tau} \right\} - \sigma_\zeta^2 \left\{ \frac{cc^*}{\kappa + \kappa^*} e^{\kappa\tau} + \frac{c^*c}{\kappa^* + \kappa} e^{\kappa^*\tau} \right\}.$$

These two terms are manifestly real-valued.

Autocovariances of an LSDE(2, 1) in Real Terms

We may note that, if $\kappa = \delta + \mathrm{i}\omega$, $\kappa^* = \delta - \mathrm{i}\omega$, and $c = \alpha + \mathrm{i}\beta$, $c^* = \alpha - \mathrm{i}\beta$, then

$$\begin{aligned}\psi(t) &= ce^{\kappa\tau} + c^* e^{\kappa^*\tau} \\ &= 2e^{\delta\tau}\{\alpha\cos(\omega\tau) - \beta\sin(\omega\tau)\}\end{aligned}$$

Within the expression for the autocovariance function $\gamma_c(\tau)$, there are

$$\begin{aligned}&\kappa\kappa^* = \delta^2 + \omega^2, \quad \kappa + \kappa^* = 2\delta \quad cc^* = \alpha^2 + \beta^2, \\ &c^2\kappa^* = (\alpha + \mathrm{i}\beta)^2(\delta - \mathrm{i}\omega) = \delta(\alpha^2 - \beta^2) + 2\alpha\beta\omega + \mathrm{i}\big\{2\alpha\beta\delta - \omega(\alpha^2 + \beta^2)\big\},\end{aligned}$$

and $(c^*)2\kappa$ can be obtained from $c^2\kappa^*$ by replacing i by $-$i. Therefore,

$$\begin{aligned}\gamma_c(\tau) = \frac{\sigma_\zeta^2}{\delta^2 + \omega^2} e^{\delta\tau} \Bigg\{ &\big\{2\alpha\beta\omega - \delta(\alpha^2 - \beta^2)\big\}\cos(\omega\tau) \\ &- \big\{2\alpha\beta\delta + \omega(\alpha^2 - \beta^2)\big\}\sin(\omega\tau)\Bigg\} - \cos(\omega\tau)\frac{(\alpha^2 + \beta^2)}{\delta}.\end{aligned}$$

Estimation of a Frequency-Limited LSDE by Impulse Invariance

The principle of impulse invariance proposes that the discrete-time transfer function should be matched to the continuous-time function by equating the ordinates of the two impulse response functions at the sample points, such that

$$\sum_j d_j \mu_j^\tau = \sum_j c_j e^{\kappa_j \tau} \quad \text{for} \quad \tau \in \{0, 1, 2, \ldots\}.$$

This can be achieved by setting $d_j = c_j$ and $\mu_j = e^{\kappa_j}$ for $j = 1, 2, \ldots, p$..

If all of the poles of the continuous-time model are real-valued, then the mapping from the continuous-time poles to the discrete-time poles is invertible, and $\kappa_j = \ln \mu_j$ for all j. A necessary restriction on the ARMA poles is that $\mu > 0$.

When the poles are complex-valued, the expression $\mu = e^\kappa$ represents a many-to-one mapping from a set of continuous-time parameters to a discrete-time parameter. Then, with $\kappa = \delta + \mathrm{i}\omega$, there is

$$\mu = e^\kappa = e^\delta e^{\mathrm{i}\omega} = e^\delta \{\cos(\omega + 2\pi n) + \mathrm{i} \sin(\omega + 2\pi n)\},$$

where n is an arbitrary integer. Therefore, a restriction is required on the range of ω, such as $\omega \in [0, \pi]$.

Estimation of the Unlimited LSDE by the Autocovariance Principle

In the case where $\zeta(t)$ is from a Wiener process, the autocovariance function of the LSDE model $\phi(D)y(t) = \theta(D)\zeta(t)$ is

$$\gamma_c(\tau) = \sigma_\zeta^2 \sum_i \Big\{ \sum_j c_i c_j \frac{-e^{\kappa_i \tau}}{\kappa_i + \kappa_j} \Big\}.$$

The roots of $\kappa_1, \ldots, \kappa_p$ of $\phi(s)$ are inferred from the roots of $\alpha(z)$ of the ARMA model. The parameters of $\theta(s)$, can be obtained from the p partial-fraction coefficients in $c = [c_1, \ldots, c_p]'$ that equate the continuous time autocovaiances with the discrete-time values $\gamma_d(\tau)$ at the sample points.

Let the equations $\gamma_c(c, \tau) = \gamma_d(\tau); \tau = 0, \ldots, p-1$ be denoted, in vector form, by $\gamma_c(c) - \gamma_d = 0$. Then, the $(r+1)$-th Newton–Raphson approximation to the solution of the equations is

$$c_{(r+1)} = c_{(r)} + [D\gamma_a(c)]^{-1}_{(r)} [\gamma_a(c) - \gamma_d]_{(r)},$$

where $D\gamma_a(c)$ is the matrix of the derivatives of γ_a in respect of c.

Once the definitive value of the vector c is available, we can solve the equations

$$c_j \prod_{i \neq j} (\kappa_j - \kappa_i) = \theta(\kappa_j); \quad j = 1, \ldots, p$$

for the coefficients $\theta_0, \ldots, \theta_{p-1}$ of $\theta(s)$.

Frequency-Limited Processes and OverSampling.

In the case of a frequency-limited discrete-time process that is supported on the entire Nyquist interval $[-\pi, \pi]$, the corresponding continuous trajectory can be created by attaching a scaled sinc function or a Dirichlet kernel to each of the discrete ordinates.

In the case of a finite data sequence, the Dirichlet function interpolation is equivalent to an ordinary Fourier interpolation.

If the sampling is of a rate in excess of the maximum data frequency, then the spectrum of the data, or its periodogram, will exhibit a dead space of zero values running from the maximum data frequency up to the Nyquist frequency of π radians per sampling interval.

Before fitting an ARMA model to frequency-limited data, it is appropriate to reconstitute the continuous trajectory before sampling it at a lower rate, so as to extend the spectral structure over the entire Nyquist interval.

The consequences of fitting an AR model to frequency-limited data are illustrated in the following slides.

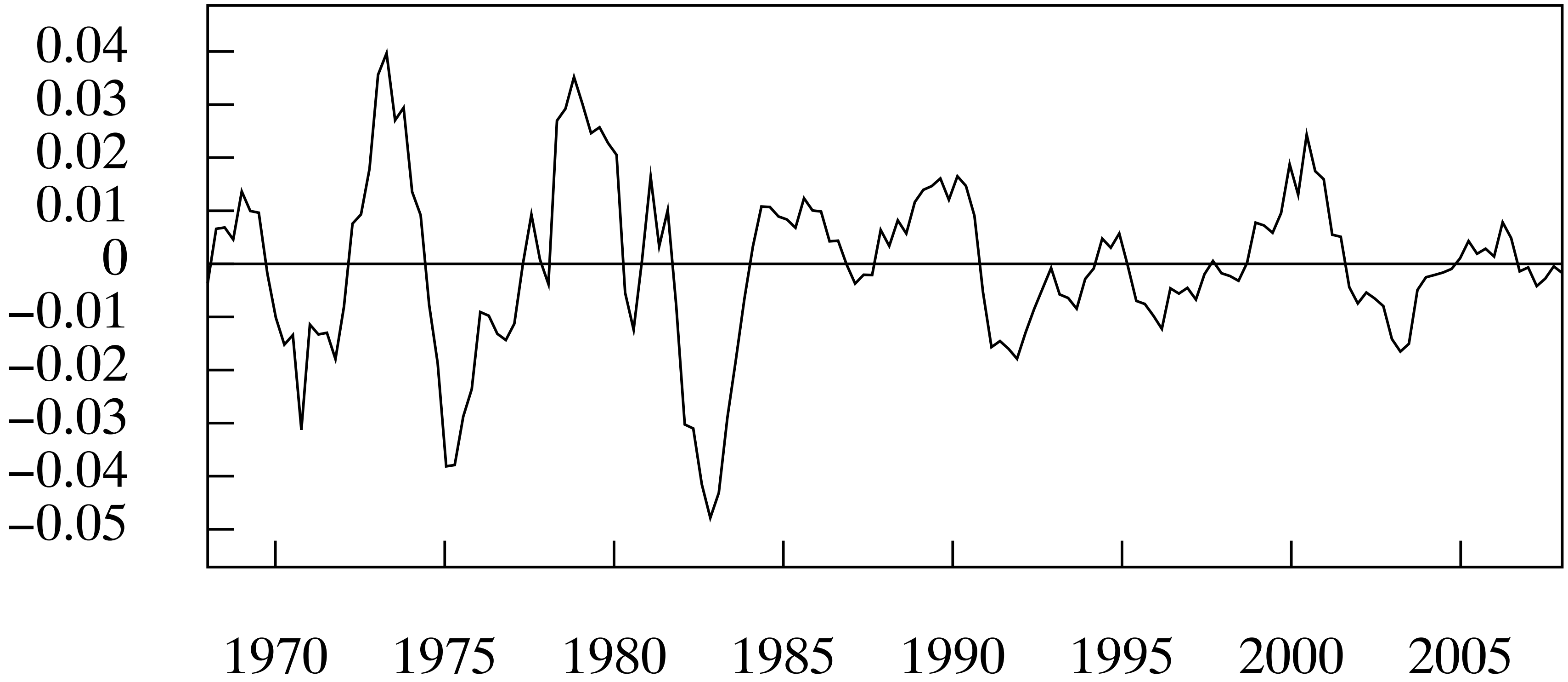

Figure 6. The deviations of the logarithmic quarterly index of real US GDP from an interpolated trend. The observations are from 1968 to 2007. The trend is determined by a Hodrick–Prescott (Leser) filter with a smoothing parameter of 1600.

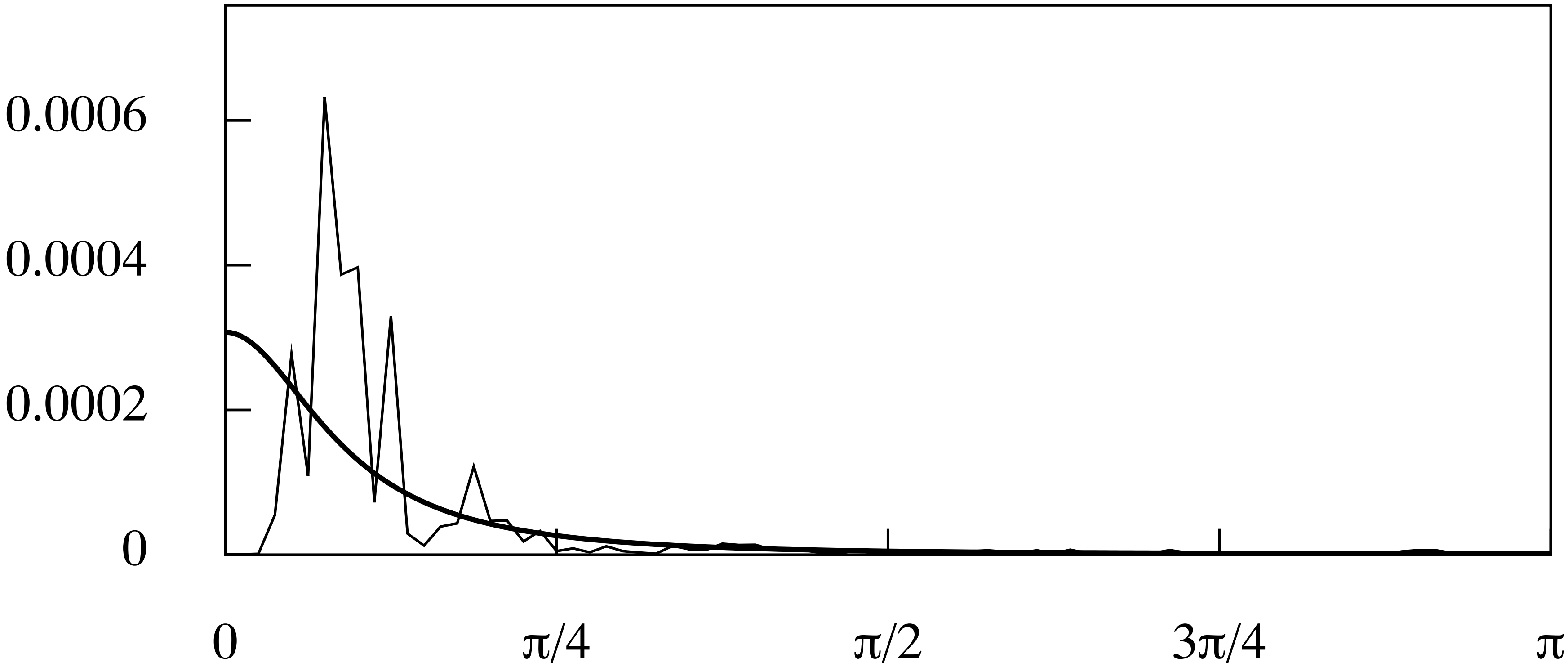

Figure 7. The periodogram of the data points of Figure 1 overlaid by the parametric spectral density function of an estimated regular AR(2) model.

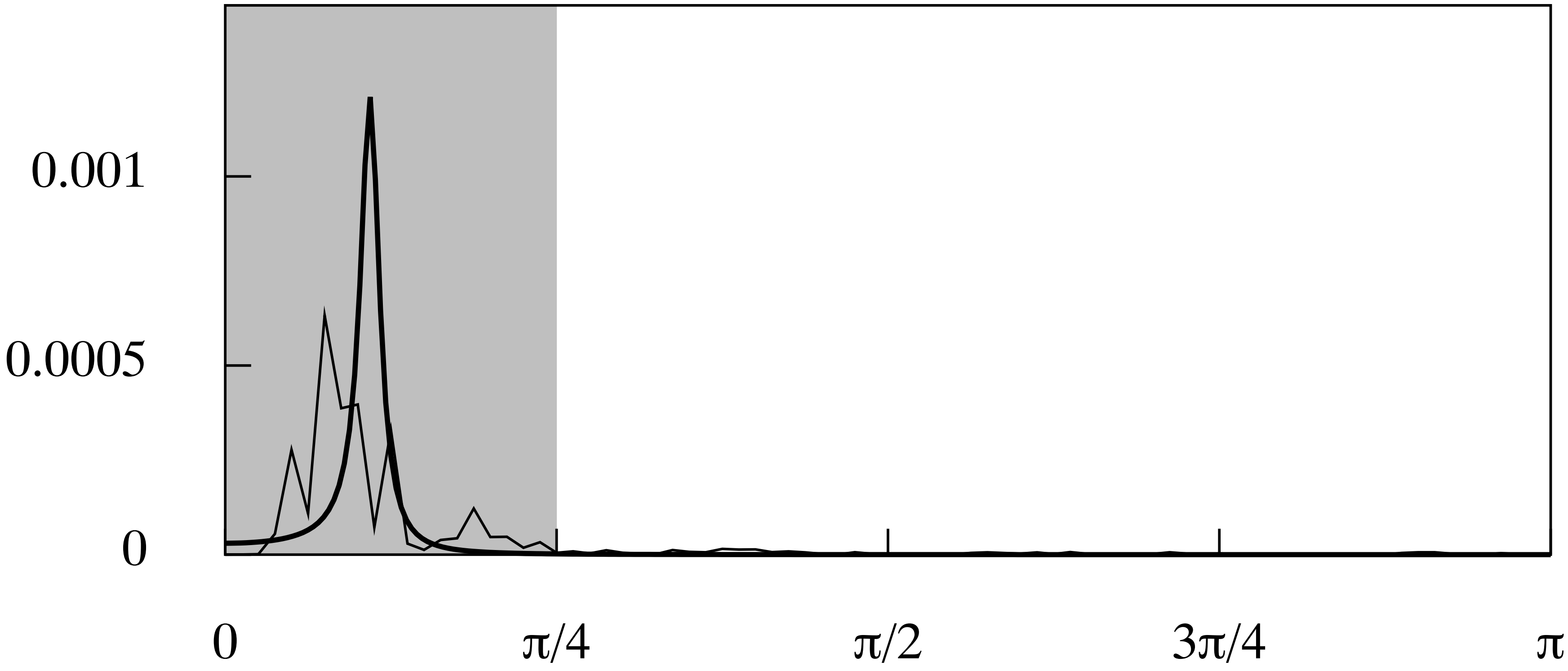

Figure 8. The periodogram of the data points of Figure 1 overlaid by the spectral density function of an AR(2) model estimated from band-limited data.

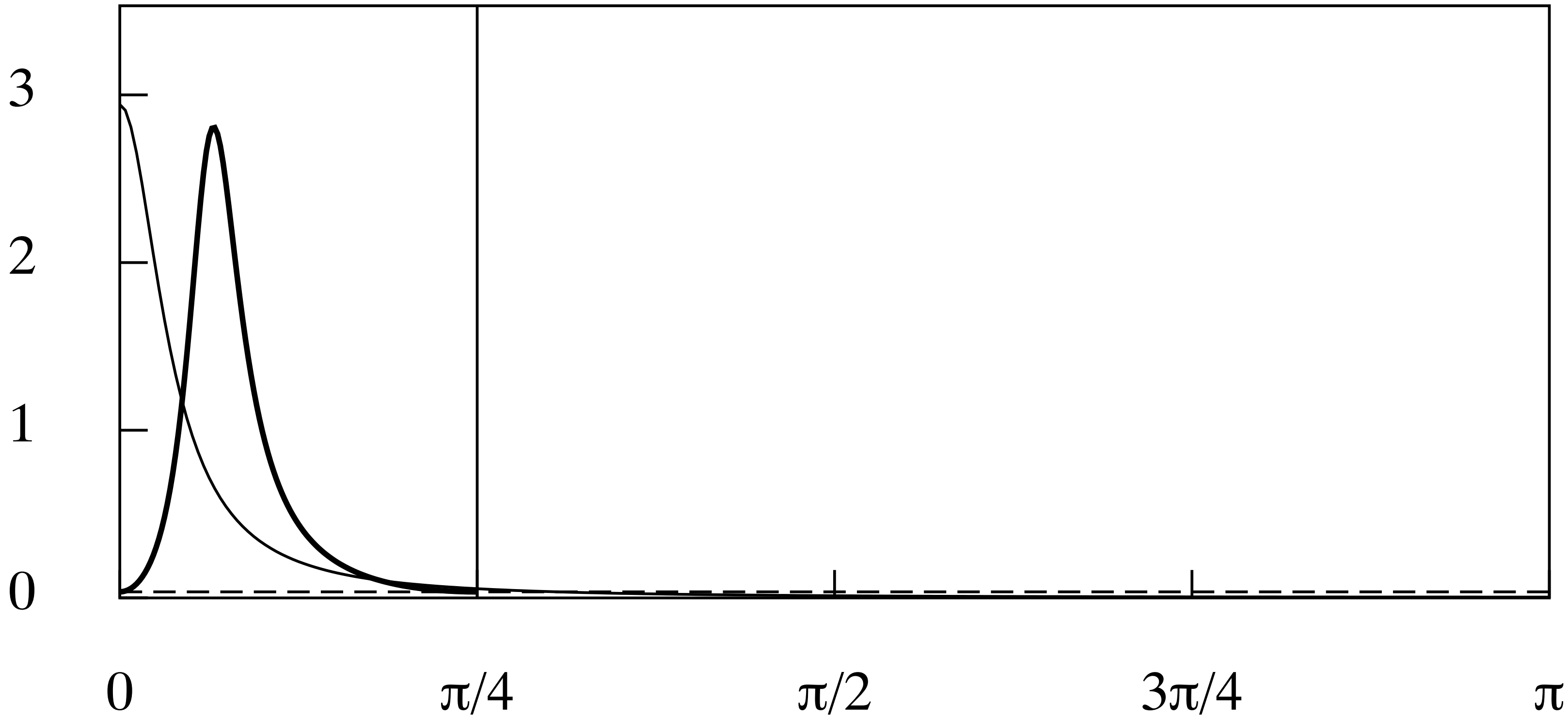

Figure 9. The parametric spectrum of the oversampled ARMA(2, 2) process, represented by a heavy line, supported on the spectrum of a white-noise contamination, together with the parametric spectrum of an AR(2) model fitted to the sampled autocovariances.

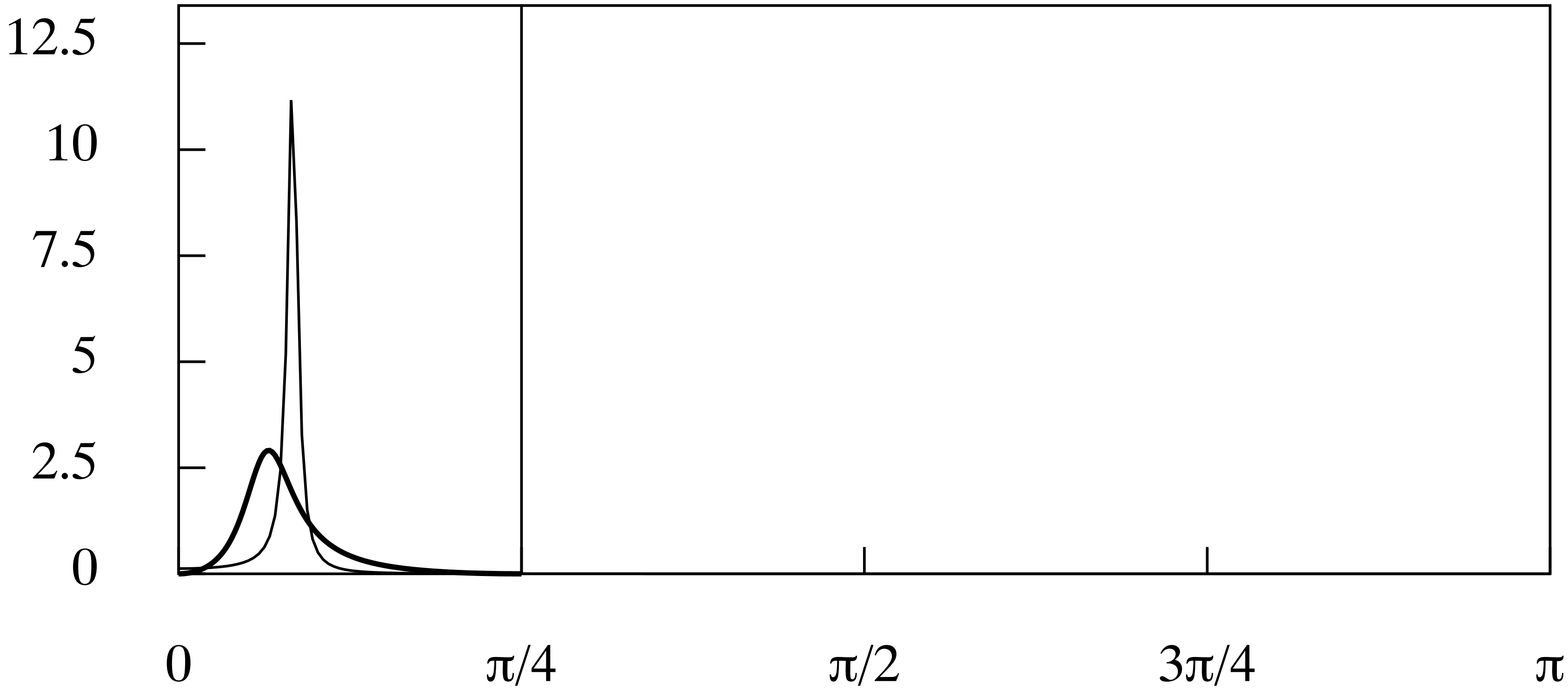

Figure 10. The parametric spectrum of an ARMA(2, 2) process, limited in frequency to π radians per period and oversampled at the rate of 4 observations per period, represented by a heavy line, together with the parametric spectrum of an AR(2) model fitted to the sampled autocovariances.

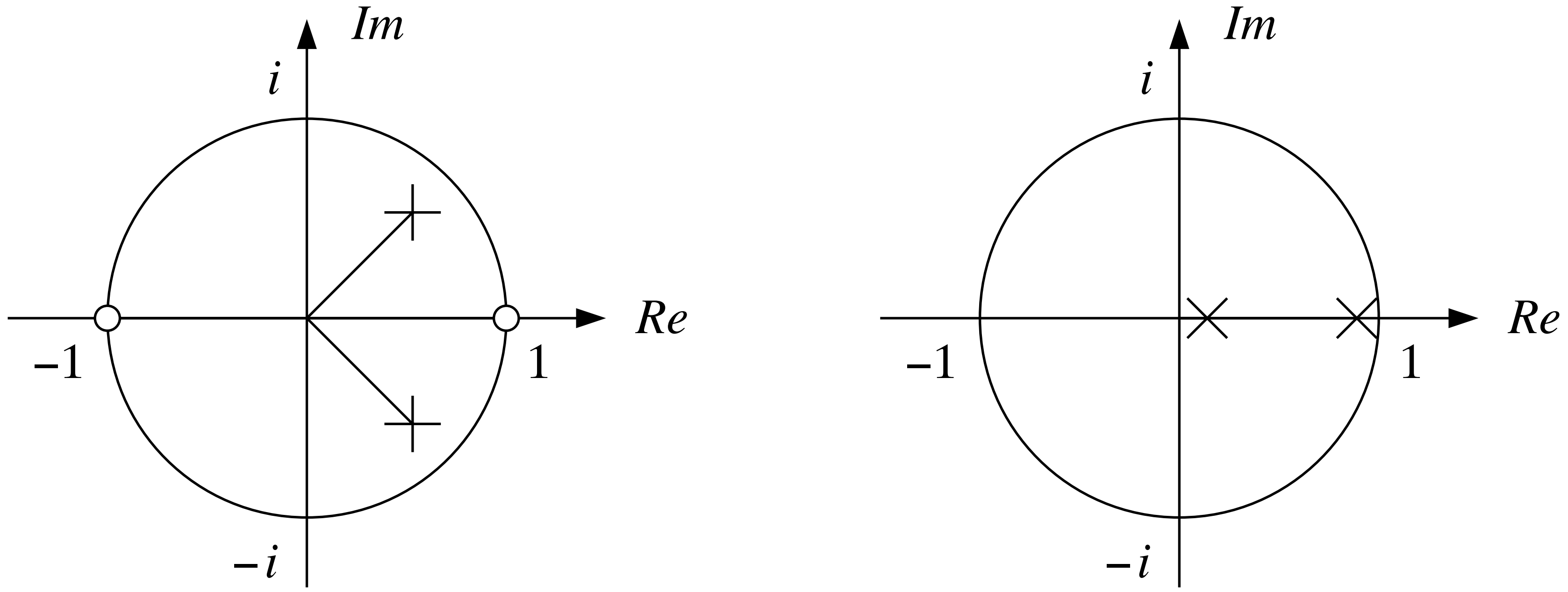

Figure. The pole-zero diagram of the ARMA(2, 2) model (left) and the diagram showing location of the poles of the AR(2) model estimated from band-limited data contaminated by white-noise errors of observation (right).

Explaining the Results

The results of these experiments can be explained by reference to the autocovariance function.

In the absense of noise contamination

When the rate of sampling is excessive, the autocovariances will be sampled at points that are too close to the origin, where the variance is to be found. Then, their values will decline at a diminished rate. The reduction in the rate of convergence is reflected in the modulus of the estimated complex roots, which understates the rate of damping.

When there is noise contamination

The variance of the white-noise errors of observation will be added to the variance of the underlying process. Nothing will be added to the adjacent sampled ordinates of autocovariance function. Therefore, the sampled autocovariances will decline at an enhanced rate.

If this rate of convergence exceeds the critical value, then there will be a transition from cyclical convergence to monotonic convergence, and the estimated autoregressive roots will be real-valued. This belies the cyclical nature of the true process.

Resampling the Data

We propose to deal with problems of oversampled data by reducing the rate of sampling.

The first step is to reconstitute a continuous trajectory from the Fourier ordinates that lie within the frequency band in question.

The second step is to sample the continuous trajectory at the rate that is precisely attuned to the highest frequency that it contains. Then, an ARMA model can be fitted in the usual way to the resampled data.
Two computer programs are available at the address

`http://www.le.ac.uk/users/dsgp1/`

The program `BLIMDOS` generates pseudo-random band-limited data (i.e. over-sampled data) from an ARMA model specified by the user. An ordinary ARMA model can be fitted to these data and the resulting biases can be assessed.

The program `OVERSAMPLE` samples the continuous autocovariance function of an ARMA model and it proceeds to infer the parameters of an ARMA model of specified orders from these sampled ordinates. In this way, it determines the probability limits of the misspecified estimators.

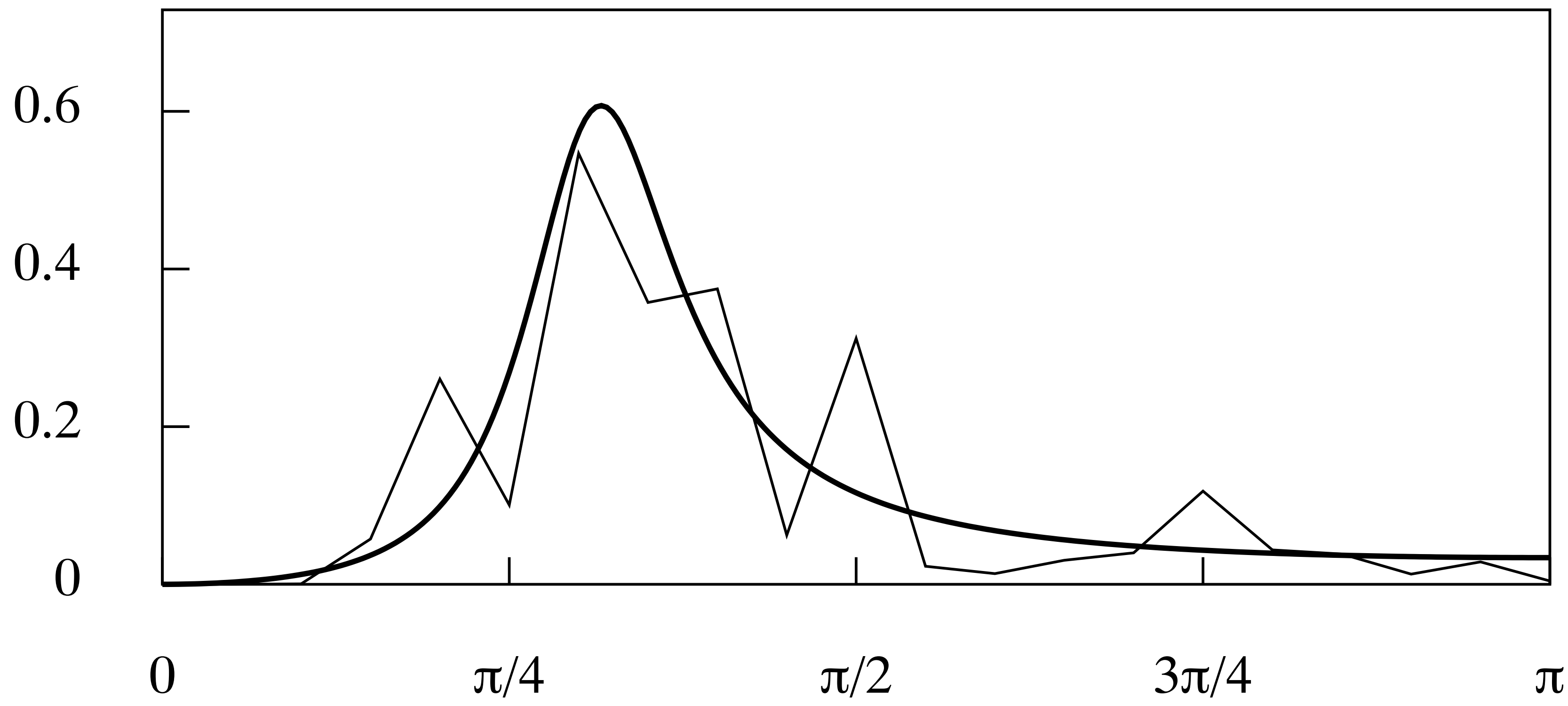

Figure 11. The periodogram of the data that have been filtered and subsampled at the rate of 1 observation in 4 overlaid by the parametric spectrum of an estimated ARMA(2, 1) model.

Characteristics of Band-Limited Processes

A function that is limited in frequency is also an analytic function. Such functions possess derivatives of all orders. Therefore the turning points of a frequency-limited business cycle trajectory can be located by finding where its derivatives are zero-valued.

A knowledge of a sufficient number of ordinates of an analytic function or of its derivatives should serve to specify the function completely. Therefore, it seems that band-limited stochastic processes should be perfectly predictable. The electrical engineers seem to be convinced of this.

To form a perfect prediction, one would need to have a denumerable infinity of sampled values, and there should be no errors of observation. The theoretical predictability of such a process depends on the infinite supports of the frequen-cybounded sinc functions with the effect that every observation comprises traces of every sinc function.

The perfect predictability of band-limited processes is an analytic fantasy of the sort that Laplace derided in a famous passage, of which the intention is often misunderstood by chaos theorists and others.

Laplacian Determinism

Laplace accepted that the events of the universe must obey the laws of nature. Nevertheless, he proposed that a statistical approach is needed in describing these events, on account of our inability to comprehend more than a few of the innumerable factors and circumstances that affect each outcome:

> "We may regard the present state of the universe as the effect of its past and the cause of its future.
>
> An intellect which, at a certain moment, would know all forces that set nature in motion, and all positions of all items of which nature is composed, if it were also vast enough to submit these data to analysis, would embrace, in a single formula, the movements of the greatest bodies of the universe and those of the tiniest atom.
>
> For such an intellect, nothing would be uncertain and the future, as much as the past, would be present before its eyes."

Pierre-Simon Laplace, (1814), *Essai Philosophique sur les Probabilités.*