ENHANCED METHODS OF SEASONAL ADJUSTMENT

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The effect of the conventional model-based methods of seasonal adjustment is to nullify the elements of the data that reside at the seasonal frequencies and to attenuate the elements at the adjacent frequencies. It may be desirable to nullify some of the adjacent elements instead of merely attenuating them. For this purpose, two alternative procedures are presented that have been implemented in a computer program.

In the first procedure, the seasonal-adjustment filter is augmented by additional poles and zeros that that are targeted at the adjacent frequencies. In the second procedure, a Fourier transform is deployed to reveal the elements of the data at all the frequencies. This allows the elements in the vicinities of the seasonal frequencies to be eliminated or attenuated at will.

In spite of the success of these procedures, the question is raised of whether the estimated trend-cycle trajectory can serve in place of the seasonally adjusted data.

Introduction

Methods for the seasonal adjustment of economic data have had a long history. The need to identify and to disregard the effects of seasonal fluctuations on economic data was perceived already by the middle of the 19th century. In their brief history of the subject, Darné, *et al.* (2018) have quoted a passage in which William Stanley Jevons (1862) evinces a wholly modern perception of the problem.

Practical methods of seasonal adjustment began to be pursued in the second decade of the 20th century. Thus, for example, Persons (2019) was proposing that economic time series could be decomposed into components that are commonly described, nowadays, as the trend, the secular cycle, the seasonal cycle and the residual fluctuations.

Modern methods of seasonal adjustment originated in the pioneering work undertaken by Julius Shiskin in the U.S. Bureau of the Census in the 1950's and 1960's, which culminated in the X-11 method, which was described by Shiskin et al. (1967). The X-11 computer program has undergone numerous improvements and modifications, leading to the X-11-ARIMA software packages of 1975 and 1988. (See Dagum 1980, 1988.) A more recent incarnation is in the X-12-ARIMA package. (See Findley *et al.* 1998.) Much of the relevant information on the method has been provided in a monograph of Ladiray and Quenneville (2001).

The seasonal adjustment that takes place nowadays in national central statistical offices is dominated by two procedures. The most enduring methods have been those of the X-11 program and its derivatives. An alternative methodology that is based on a model of a seasonal economic process has been provided by the SEATS-TRAMO program, which was originated by Maravall and Gómes (1992).

Both the X-12 program and the SEATS-TRAMO program are incorporated in the Demetra program of the Bureau of the Census, also refered to as the X-13ARIMA-SEATS program. Its purpose is to allow for the cross checking of the results of the alternative procedures.

The model-based methodology works well when applied to data to which the model fits well. It is accompanied by some sophisticated methods that are designed to deal with data anomalies that are the effects of a shifting calendar.

Seasonal fluctuations do not always manifest a metronomic regularity, and their patterns are liable to evolve over time. Therefore, to represent them well, it is necessary to take account of more than just the elements of the data that are to be found at the fundamental seasonal frequency and its harmonics, described hereafter as the seasonal frequencies. The adjacent elements should also be entailed in the representations.

The conventional methods of seasonal adjustment do take this into account, but only to a limited extent. As has been observed by McElroy and Roy (2017) and by others, there are cases where some of the elements of the seasonal fluctuations evade these methods. This paper proposes methods that might be relied on to overcome such difficulties. Two alternative methods are proposed.

In the first method, described as the time-domain method, some modifications to a model-based procedure are pursued. The model in question has been intended to give rise to filters that mimic the seasonal-adjustment filters of the conventional methods. The task of estimating the model is avoided and the parameters of the model are chosen by the user in view of the intended features of the resulting filter.

To accommodate additional seasonal elements, a method of frequency modulation is proposed that increases the width of the stop bands within the filter's frequency response that are intended eliminate the seasonal fluctuations.

The second method, which is the frequency-domain method, is intended to achieve a similar outcome by working directly with the elements of the Fourier decomposition of the data. Thus, the frequency-domain elements of the data may be preserved or attenuated or eliminated at will.

Whereas this method is able impose whatever frequency response has been generated by the time-domain method, it has the ability to impose an arbitrary frequency response. This gives it superior flexibility.

Linear Filtering

A linear filtering operation applied to a discrete-time signal $y(t) = \{y_t; t = \pm 1, \pm 2, \ldots\}$, can be represented by the equation

$$x(t) = \sum_j \psi_j y(t-j). \tag{1}$$

To assist in the algebraic manipulation of such equations, the data sequence $y(t)$, its filtered version $x(t)$ and the sequence of filter coefficients $\{\psi_j\}$ may be converted to power series or polynomials. By associating z^t to each element x_t and by summing the sequence, the following equation is derived:

$$\sum_t x_t z^t = \sum_t \Big\{ \sum_j \psi_j y_{t-j} \Big\} z^t \quad \text{or} \quad x(z) = \Psi(z) y(z), \tag{2}$$

where

$$x(z)=\sum_t x_t z^t, \quad y(z)=\sum_t y_t z^t \quad \text{and} \quad \Psi(z)=\sum_j \psi_j z^j. \tag{3}$$

The convolution operation of equation (1) becomes an operation of polynomial multiplication in equation (2). The z-transform $\Psi(z)$ of the filter coefficients is described as the transfer function of the filter.

In many practical applications, the transfer function is liable to be a symmetric rational function of the form

$$\Psi(z)=\frac{\Theta(z)}{\Phi(z)}=\frac{N(z^{-1})N(z)}{\Delta(z^{-1})\Delta(z)}, \tag{4}$$

where $N(z)=\nu_0+\nu z+\cdots+\nu_q z^q$ may be described as the numerator polynomial and $\Delta(z)=\delta_0+\delta z+\cdots+\delta_q z^p$ may be describe as the denominator polynomial. A rational filter has the advantage of a flexibility that affords superior approximations to ideal specifications. The consequence of symmetry is that the filter has no phase effects, which is to say that it imposes no delays on the processed data.

The expansion of the rational function will give rise to a doubly-infinite sequence of coefficients. Therefore, the filter cannot be applied directly in the manner of equation (1), unless one is prepared to truncate the sequence. Instead, it may be applied in two passes running through the data in opposite directions. These processes can be represented by the equations

$$\sum_{j=0}^{p}\delta_j q_{t-j}=\sum_{j=0}^{q}\nu_j y_{t-j} \qquad \text{and} \qquad \sum_{j=0}^{p}\delta_j x_{t+j}=\sum_{j=0}^{q}\nu_j q_{t+j}, \tag{5}$$

of which the z-transforms are

$$\Delta(z)q(z)=N(z)y(z) \qquad \text{and} \qquad \Delta(z^{-1})x(z)=N(z^{-1})q(z). \tag{6}$$

Such filters represent a most effective way of processing economic data in pursuance of a wide range of objectives.

The frequency response function of the filter shows the degree to which it alters the amplitudes of the elements of the data at the various frequencies in the interval $[-\pi,\pi]$. The function is obtained by setting $z=\exp(-\mathrm{i}\omega)$ within the transfer function $\Psi(z)$, which places it on the unit circle in the complex plane, and by having it travel around the circle by running ω from $-\pi$ to π or from 0 to 2π

Comb Filters

The model-based methods of seasonal adjustment embody comb filters that serve to eliminate the elements of the data at the seasonal frequencies and to attenuate the elements at the adjacent frequencies. The attenuation diminishes as the distance from the seasonal frequencies increases.

The nullification of the seasonal elements is achieved by the zeros of the filter, which are the roots of its numerator polynomial. These are located on the unit circle at angles that correspond to the seasonal frequencies. The effects of these zeros at other frequencies is limited by the presence of the poles of the filter that lie on the same axes or radii as the zeros and that are close to the unit circle. At frequencies that are remote from the seasonal frequencies, the effects of the poles and the zeros are largely cancelled.

The zeros of the comb filters that are employed in seasonal adjustment are some of the roots of unity, which are the solutions of the equation $1 = z^s$, where s is the number of observations within a year. For quarterly data, there is $s = 4$ and, for monthly data, there is $s = 12$. The roots are located on the unit circle; and the angles to the horizontal of their radii are $\omega_j = 2\pi j/s; j = 1, 2, \ldots, s$.

The poles that can accompany the zeros are provided by the solution to the equation $1 = \rho^s z^s$, where $\rho < 1$ is close to unity. These are the roots of the denominator polynomial of the filter. The poles take the values $\rho\exp(\mathrm{i}2\pi j/s); j = 1, \ldots, s$, which is to say that they lie on a circle in the complex plane of radius ρ^{-1}.

In practice, the filters for seasonal adjustment do not include the poles or zeros at zero frequency, since their presense would affect the trend component of the filtererd data. These can be eliminated from the numerator polynomial by dividing by $1 - z$ and from the denominator polynomial by dividing by $1 - \rho z$. The resulting polynomials are

$$\Sigma(z) = \frac{1 - z^s}{1 - z} = 1 + z + z^2 + \cdots + z^{s-1} \tag{7}$$

and

$$P(z) = \Sigma(\rho z) = \frac{1 - \rho^s z^s}{1 - \rho z} = 1 + \rho z + \rho^2 z^2 + \cdots + \rho^{s-1} z^{s-1}. \tag{8}$$

These polynomials may be compounded from the quadratic factors that correspond to the pairs of conjugate complex roots and from a linear factor that corresponds to the limiting Nyquist frequency. The jth quadratic factor is

$$\begin{aligned}(1 - \rho_j e^{\mathrm{i}\omega_j} z)(1 - \rho_j e^{-\mathrm{i}\omega_j} z) &= 1 - \rho_j(e^{\mathrm{i}\omega_j} + e^{-\mathrm{i}\omega_j})z + \rho_j^2 z^2 \\ &= 1 - 2\rho_j \cos(\omega_j)z + \rho_j^2 z^2,\end{aligned} \tag{9}$$

where $j = 1, 2, \ldots, (s-2)/2$ and where the second equality is by virtue of $\cos(\omega_j) = \{\exp(\mathrm{i}\omega_j) + \exp(-\mathrm{i}\omega)j)\}/2$. The linear factor is $1 + \rho_{s/2} z$.

In the case of $\Sigma(z)$, there is $\rho_j = 1$ for all j and, in the case of $P(z) = \Sigma(\rho z)$, there is $\rho_j = \rho$ for all j. However, an additional flexibility would come from allowing ρ to vary with the index j. Such a generalisation has been pursued by Findley and Martin (2006).

Given that it is symmetric, the rational filter can be cast in the form of

$$\Psi(z) = \frac{\Sigma(z^{-1})\Sigma(z)}{P(z^{-1})P(z)} = \frac{\theta_0 + \theta_1(z + z^{-1}) + \cdots + \theta_{s-1}(z^{s-1} + z^{1-s})}{\phi_0 + \phi_1(z + z^{-1}) + \cdots + \phi_{s-1}(z^{s-1} + z^{1-s})}. \tag{10}$$

Setting $z = \exp(-\mathrm{i}\omega)$ leads to an expression for the frequency response function in terms of cosines:

$$\Psi(\omega) = \frac{\Theta(\omega)}{\Phi(\omega)} = \frac{\theta_0 + 2\{\theta_1\cos(\omega) + \cdots + \theta_{s-1}\cos([s-1]\omega)\}}{\phi_0 + 2\{\phi_1\cos(\omega) + \cdots + \phi_{s-1}\cos([s-1]\omega)\}}. \tag{11}$$

Here, the abbreviation $\Theta(\omega)$ has been used in place of $\Theta\{\exp(-\mathrm{i}\omega)\} = \Sigma\{\exp(\mathrm{i}\omega)\}\Sigma(\exp\{-\mathrm{i}\omega)\}$ and likewise for $\Phi(\omega)$,

As it stands, the filter will have a gain that is liable to exceed unity at certain frequencies. A natural requirement is that the gain should be be unity at the zero frequensy. Thus can be achieved by diving the numerator $\Theta(z)$ by s^2, which is the sum of its coefficients, and by dividing the denominator $\Phi(z)$ by $\phi_0 + 2\sum_{j=1}^{s-1}\phi_j$.

Given that it is compounded from cosine functions, the frequency response function is symmetric with respect to the zero frequency, such that $\Psi(-\omega) = \Psi(\omega)$. Therefore, it is conventional to represent the frequency response function by plotting it only over the interval $[0, \pi]$ of the positive frequencies.

The Wiener–Kolmogorov Filter

Although a specification has been provided already for a comb filter that will achieve a seasonal adjustment, it is appropriate to develop a similar filter within the framework of an heuristic statistical model by following the Wiener–Kolmogorov methodology of signal extraction.

Such an approach generates minimum mean-square error estimates of the sought-after components on the condition that the model is correctly specified. In that case, the estimates are the expectations of the components that are conditional on the available data. Wiener–Kolmogorov approach also leads to a method for treating trending data sequences of a limited duration.

The model in question is of a process that has no trend is its level. It may be imagined that the trend has already been removed from the original data by fitting a polynomial function or by a more sophisticated method of a sort that will be outlined later. Once the seasonal fluctuations have been removed from the non-trended process, the residue may be added back to the trend function to create the seasonally-adjusted sequence.

Alternatively, it may be imagined that the trend has been removed by differencing the data. After the extraction of the seasonal fluctuations, the filtered differenced data may be re-inflated by a process of summation, which should serve to generate the seasonally-adjusted data. Amongst the processes that will be reduced to a white-noise sequence by double differencing is a drifting random walk, which will often prove to be an adequate model for an economic data sequence.

The heuristic model of the detrended data may be represented in a z-transform notation by

$$\begin{aligned} y(z) &= \frac{P(z)}{\Sigma(z)}\nu(z) + \eta(z) \\ &= \xi(z) + \eta(z). \end{aligned} \tag{12}$$

Here, $\xi(z)$ represents the seasonal fluctuations, whereas $\eta(z)$ represents whatever other motions may be present in the detrended data. Both $\nu(z)$ and $\eta(z)$ are represented by mutually independent white-noise processes with variances of σ_ν^2 and σ_η^2, respectively.

The presence of complex roots of unit modulus within the polynomial $\Sigma(z)$ implies that the process is nonstationary in amplitude. It may be reduced to stationarity by multiplying throughout by $\Sigma(z)$ to give

$$\begin{aligned}\Sigma(z)y(z) &= P(z)\nu(z) + \Sigma(z)\eta(z) \\ &= \delta(z) + \kappa(z) = g(z).\end{aligned} \tag{13}$$

The z-tranform of the Wiener–Kolmogorov filter that serves, equally, to extract $\eta(z)$ from $y(z)$ and $\Sigma(z)\eta(z)$ from $\Sigma(z)y(z)$ is

$$\beta(z) = \frac{\sigma_\eta^2 \Sigma(z^{-1})\Sigma(z)}{\sigma_\eta^2 \Sigma(z^{-1})\Sigma(z) + \sigma_\nu^2 P(z^{-1})P(z)} = \frac{\Sigma(z^{-1})\Sigma(z)}{\Pi(z^{-1})\Pi(z)}, \tag{14}$$

where

$$\Pi(z^{-1})\Pi(z) = \Sigma(z^{-1})\Sigma(z) + \lambda P(z^{-1})P(z) \quad \text{with} \quad \lambda = \frac{\sigma_\nu^2}{\sigma_\eta^2}. \tag{15}$$

This is the comb filter, with poles and zeros on the same radii as those of the filter specified previously. However, the poles are at distances from the unit circle that vary with the index j.

The inclusion of the term $\sigma_\eta^2 \Sigma(z^{-1})\Sigma(z)$ in the denominator of (14) means that the gain of the Wiener–Kolmogorov filter cannot exceed unity. This is in contrast to the simple comb filter where the gain is liable to exceed unity at certain frequencies. However, to ensue that unity is attained at the zero frequency, it is appropriate to normalise the filter by dividing the coefficients of the numerator and denominator by their respective sums.

The Finite-Sample Wiener–Kolmogorov Filter

To derive the finite-sample version of the Wiener–Kolmogorov filter, consider a vector $y = [y_0, y_1, \ldots, y_{T-1}]'$ of T values drawn from the process represented by $y(z)$. In accordance with equation (12), the vector may be decomposed as

$$y = \xi + \eta. \tag{16}$$

To create a vector of a stable amplitude, this vector must be transformed by a matrix $\Sigma_s = \Sigma(L_T)$ of order T, which is the finite-sample analogue of the operator $\Sigma(z)$. This matrix is derived by replacing the argument z by the matrix lag operator $L_T = [e_1, \ldots, e_{T-1}, 0]$ of order T, which is derived from the identity matrix $I_T = [e_0, e_1, \ldots, e_{T-1}]'$ by deleting the leading column and by adding a column of zeros to the end of the array.

When applying the matrix operator to the vector y, the first s elements of the product, which are in g_*, are liable to be discarded:

$$\Sigma(L_T)y = \begin{bmatrix} S_*' \\ S' \end{bmatrix} y = \begin{bmatrix} g_* \\ g \end{bmatrix}. \tag{17}$$

Here S', is a matrix of order $(T-s) \times T$ of which the jth row contains the s unit coefficients of $S(z)$ preceeded by $j-1$ zeros and followed by zeros. The matrix S_*

of order $s \times T$ contains a leading lower-triangular matrix filled with units and a following matrix of order $s \times (T - s)$ full of zeros.

In common with $\Sigma(L_T)$, the finite-sample analogue of the operator $P(L_T)$ has a Toeplitz structure as follows:

$$P(L_T) = \left[\begin{array}{cccccccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \rho & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ \rho^{s-2} & \rho^{s-3} & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ \hline \rho^{s-1} & \rho^{s-2} & \cdots & \rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & \rho^{s-1} & \cdots & \rho^2 & \rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho^{s-1} & \rho^{s-2} & \rho^{s-3} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \rho^{s-1} & \rho^{s-2} & \cdots & \rho & 1 \end{array}\right] = \begin{bmatrix} R'_* \\ R' \end{bmatrix}. \tag{18}$$

Thus, $P(L_T)$ becomes $\Sigma(L_T)$ when $\rho = 1$.

Applying S' to the equation $y = \xi + \eta$, representing the seasonally fluctuating data, gives

$$\begin{aligned} S'y &= R'\nu + S'\eta \\ &= \delta + \kappa = g. \end{aligned} \tag{19}$$

This is just a segment of $T - s$ elements drawn from the process represented by equation (13).

The expectations and the dispersion matrices of the component vectors g are

$$\begin{aligned} E(\delta) &= 0, \qquad D(\delta) = \sigma_\nu^2 R'R, \\ E(\kappa) &= 0, \qquad D(\kappa) = \sigma_\nu^2 S'S. \end{aligned} \tag{20}$$

The difficulty of estimating the vector $\xi = y-\eta$ of seasonal fluctuations directly is that some starting values or initial conditions are required in order to define the value at time $t = 0$. However, since η is from a stationary mean-zero process, it requires only zero-valued initial conditions. Therefore, the starting-value problem can be circumvented by concentrating on the estimation of η, wherafter an estimate of $\xi = y - \eta$ is readily available. The estimates of ξ and η will be denoted by the roman letters x and h respectively.

The conditional expectation of η, given the transformed data $g = S'y$, is provided by the formula

$$\begin{aligned} h = E(\eta|g) &= E(\eta) + C(\eta, g)D^{-1}(g)\{g - E(g)\} \\ &= C(\eta, g)D^{-1}(g)g, \end{aligned} \tag{21}$$

where the second equality follows in view of the zero-valued expectations of η and g. Within this expression, there are

$$D(g) = \sigma_\nu^2 R'R + \sigma_\eta^2 S'S \quad \text{and} \quad C(\eta, g) = \sigma_\eta^2 S. \tag{22}$$

Putting these details into (21) gives the following estimate of η:

$$\begin{aligned} h &= \sigma_\eta^2 S(\sigma_\nu^2 R'R + \sigma_\eta^2 S'S)^{-1}S'y \\ &= S(S'S + \lambda R'R)^{-1}S'y, \end{aligned} \tag{23}$$

whence

$$\begin{aligned} x = E(\xi|g) &= y - E(\eta|g) = y - h \\ &= \{I - S(S'S + \lambda R'R)^{-1}S'\}y. \end{aligned} \tag{24}$$

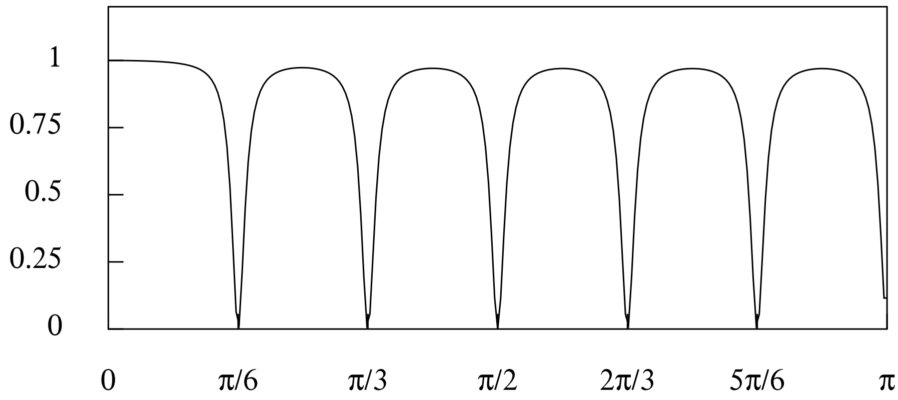

Figure 1. The frequency response function of the ordinary seasonal adjustment filter for monthly data.

Figure 1 shows the ordinary seasonal adjustment filter for monthly data when the smoothing parameters is $\lambda = 0.5$ and the pole parameter is $\rho = 0.8$. There is a complete nullification of the elements at the seasonal frequencies; and those at the adjacent frequencies are attenuated to an extent that diminishes rapidly as their distance from the seasonal frequencies increases.

Widening the Seasonal Stopbands

To increase the attenuation of the elements of the data that are adjacent to the seasonal frequencies, one can reduce the value of ρ within the polynomial $P(z)$. This will draw the poles away from the unit circle, with an effect that can be seen by comparing the two sides of Figure 2.

It may be required to impose a greater attenuation on the adjacent elements than can be achieved by reductions in the value of ρ, and it may be desirable to confine this effect more narrowly to the vicinities of the seasonal frequencies.

For this purpose, it might be appropriate to apply the seasonal-adjustment filter twice or more in succession and with zeros that are displaced from the seasonal frequencies by small angles. A twofold filter with equal displacements on either side of the seasonal frequencies could take the form of

$$\Psi_\zeta(\omega) = \Psi(\omega - \zeta)\Psi(\omega + \zeta), \tag{25}$$

where ζ is the angle of the displacement. It should be observed that $\cos(\pi + \zeta) = \cos(\pi - \zeta)$. Thus, the same factor is present in both $\Psi(\omega - \zeta)$ and $\Psi(\omega + \zeta)$. To avoid the duplication, it is reasonable to exclude the factor from the first of these filters.

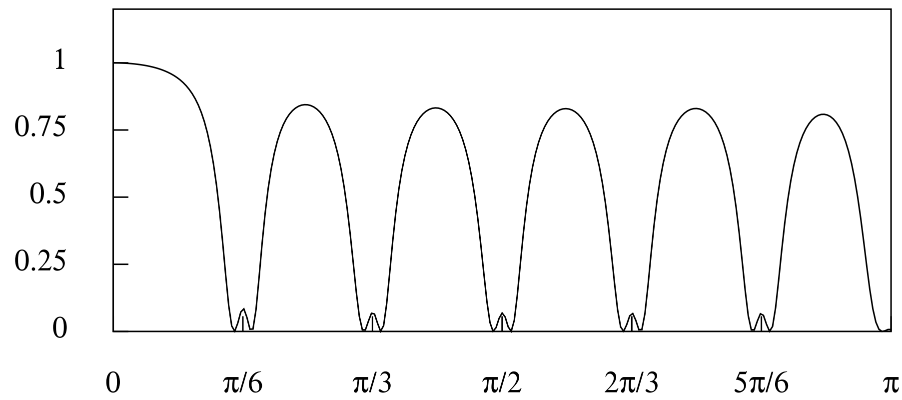

Figure 2. The frequency response function of the double seasonal adjustment filter for monthly data with offsets of 2 degrees.

Figure 2 shows the frequency response of the double filter in which the offsets are $\pm\zeta = \pm 0.2$ degrees (0.0349 radians). The lack of zeros at the seasonal frequencies allows a small amount of leakage to occur, which increases with the size of the offsets. Given the likely prominence of the elements of the data at the seasonal frequencies, this leakage is liable to prove problematic.

To overcome the leakage of the double filter, it is possible to combine the standard filter with the two offset filters to create a triple filter. The first and the primary filter will have its poles and zeros at exactly the seasonal frequencies. The second and the third of the filters will have theirs poles and zeros offset to the left and to the right, respectively. However, it may be desirable to apply differing offsets relative to some or all of the seasonal frequencies.

Thus, if it were required to place additional poles and zeros on either side of the frequency ω_j, then it would be appropriate to compound the denominator polynomial $N(z)$ with the factors

$$1 - 2\rho\cos(\omega_j + \zeta_1)z + \rho z^2 \quad \text{and} \quad 1 - 2\rho\cos(\omega_j - \zeta_2)z + \rho z^2 \tag{26}$$

and to compound the numerator polynomial $\Delta(z)$ with a similar polynomial, but with $\rho = 1$.

The appropriate displacements can be determined with reference to the periodogram of the seasonal data after their trend has been removed. This will indicate which of the data elements adjacent to the seasonal elements should be taken into account, to be eliminated or attenuated.

The frequency response of such a triple filter is illustrated in Figure 3. Here, the values of $\lambda = 0.5$ and $\rho = 0.8$, which have characterised the previous filters, are retained. However, an offset of 3 degrees (0.0524 radians) has bee applied on either side of each of the seasonal frequencies .

A problem with the frequency response of the triple filter is that its values at the midpoints between the seasonal frequencies are significantly less that unity. This conflicts with the intention of preserving the elements of the data at these points and in the vicinities thereof. A similar but a less severe problem also arises with the double filter. Also, the problem of the end-of-sample effects is accentuated by applying three filters in succession. These problems can be overcome by operating in the frequency domain.

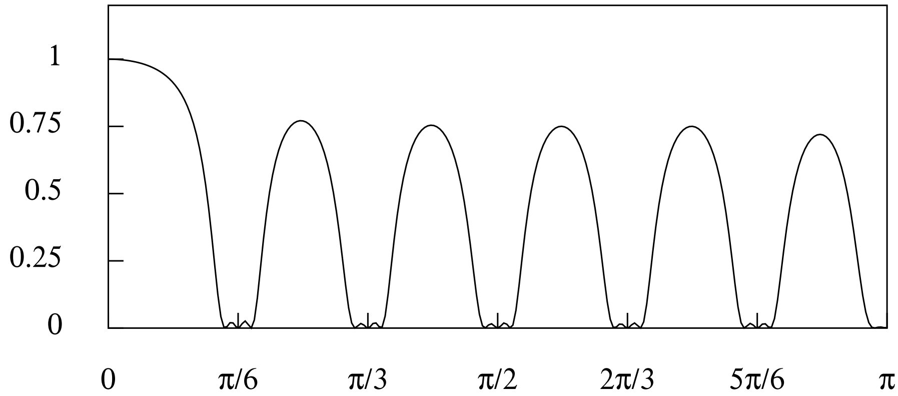

Figure 3. The frequency response function of the triple seasonal adjustment filter for monthly data with offsets of 3 degrees.

The Frequency-Domain Filter

An alternative way of applying a filter is to operate on the frequency-domain or Fourier ordinates of the data, which are supplied by the discrete Fourier transform. The relationships between the data sequence $\{y_t; t = 0, 1, \ldots, T-1\}$ and the Fourier ordinates $\{\zeta_j; j = 0, 1, \ldots, T-1\}$ is represented by

$$y_t = \sum_{j=0}^{T-1} \zeta_j e^{\mathrm{i}\omega_j t} \longleftrightarrow \zeta_j = \frac{1}{T}\sum_{t=0}^{T-1} y_t e^{\mathrm{i}\omega_j t}, \qquad \text{with} \qquad \omega_j = \frac{2\pi j}{T}. \tag{27}$$

The first of these equations, which depicts the inverse Fourier transform, represents the Fourier synthesis of the data, whereas the second equation depicts the direct Fourier transform of the data.

The data can also be expressed in terms of a set of mutually orthogonal trigonometric functions:

$$y_t = \sum_{j=0}^{[T/2]} (\alpha_j \cos \omega_j t + \beta_j \sin \omega_j t), \tag{28}$$

where $[T/2]$ is the quotient (i.e, the integral part) of $T/2$. The coefficients of this equation are

$$\alpha_j = \zeta_j + \zeta_{T-j} \qquad \text{and} \qquad \mathrm{i}\beta_j = \zeta_{T-j} - \zeta_j, \tag{29}$$

whereas, according to Euler's equations, there are

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{-\mathrm{i}}{2}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}) = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}). \tag{30}$$

The coefficients of equation (28) are obtained by projecting the data onto the trigonometrical functions, which gives

$$\alpha_0 = \frac{1}{T}\sum_t y_t = \bar{y}, \quad \alpha_j = \frac{2}{T}\sum_t y_t \cos\omega_j t \quad \text{and} \quad \beta_j = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{31}$$

The seasonal-adjustment filter can be implemented in the frequency domain by multiplying the Fourier ordinates or the coefficients of the trigonometric function by factors derived from the appropriate frequency response function. The results are carried back to the time domain by an inverse Fourier transform or, equally, via a trigonometric synthesis.

In choosing an appropriate frequency response, there is more flexibility in the frequency-domain approach than there is in the time-domain approach

References

Darné, O., L. Ferrara and D. Ladiray, (2018), A Brief History of Seasonal Adjustment Methods and Software Tools, in Mazzi, G.L., D. Ladiray, and D.A. Rieser, (eds.), Handbook on Seasonal Adjustment 2018 edition, Eurostat: Publications Office of the European Union, Luxembourg.

Findley, D.F., and D.K. Martin, (2006), Frequency Domain Analyses of SEATS and X-11/12-ARIMA Seasonal Adjustment Filters for Short and Moderate-Length Time Series, *Journal of Official Statistics,* 22, 1–34.

Maravall, A., and V. Gómez, (1992), Signal Extraction in ARIMA Time Series: Program SEATS, Working Paper ECO No 92-65, Department of Economics, European University Institute.

Maravall, A., and G. Caporello (2004), Program TSW: Revised Reference Manual. Technical Report, Research Department, Bank of Spain.

Mazzi, G.L., D. Ladiray, and D.A. Rieser, (2018), Handbook on Seasonal Adjustment 2018 Edition, Eurostat: Publications Office of the European Union, Luxembourg.

McElroy, T., and A. Roy, (2017), Detection of Seasonality in the Frequency Domain, Center for Statistical Research & Methodology Research and Methodology Directorate U.S. Census Bureau, Washington.

Persons W.M. (1919), Indices of General Business Conditions, Review of Economic Statistics, 1, 111-205.

Shiskin J. (1957), Electronic Computers and Business Indicators, Occasional Paper No 57, National Bureau of Economic Research, New York.

Shiskin, J., A.H. Young, and J.C. Musgrave, (1967), The X-11 Variant of the Census Method II Seasonal Adjustment, Technical Paper No. 15, Bureau of the Census, U.S. Department of Commerce.

U.S. Census Bureau, (2013), X-13ARIMA-SEATS Reference Manual, Time Series Research Staff, Statistical Research Division., U.S. Bureau of the Census, http://www.census.gov/srd/www/x13as/.