5. THE CLASSES OF FOURIER TRANSFORMS

There are four classes of Fourier transform, which are represented in the following table. So far, we have concentrated on the discrete Fourier transform.

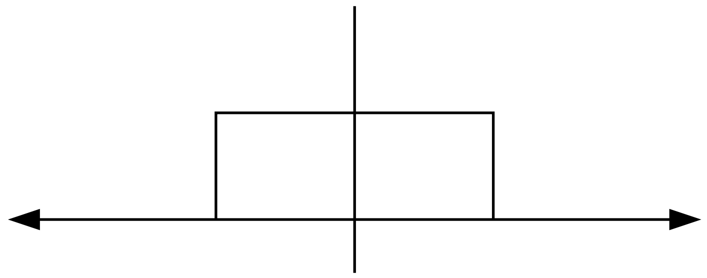
Table 1. The classes of Fourier transforms*

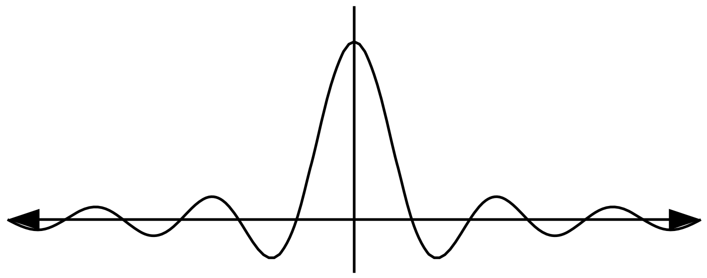
	Periodic	**Aperiodic**
Continuous	Discrete aperiodic *Fourier series*	Continuous aperiodic *Fourier integral*
Discrete	Discrete periodic *Discrete FT*	Continuous periodic *Discrete-time FT*

* The class of the Fourier transform depends upon the nature of the function which is transformed. This function may be discrete or continuous and it may be periodic or aperiodic. The names and the natures of corresponding transforms are shown in the cells of the table.

(i) The Fourier integral:

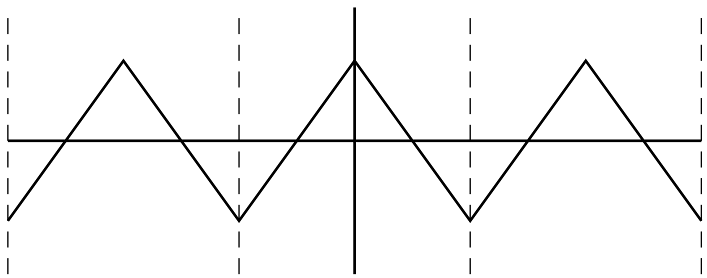
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega)e^{\mathrm{i}\omega t}d\omega \quad \longleftrightarrow \quad \xi(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}\omega t}dt$$

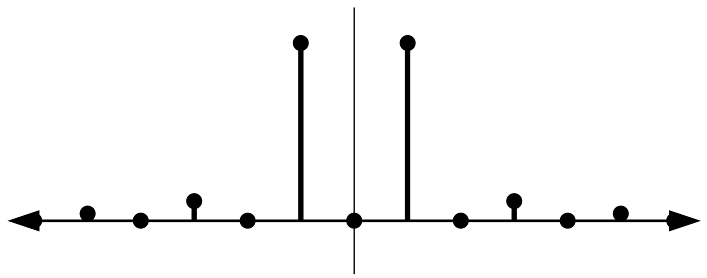




(ii) The classical Fourier series:

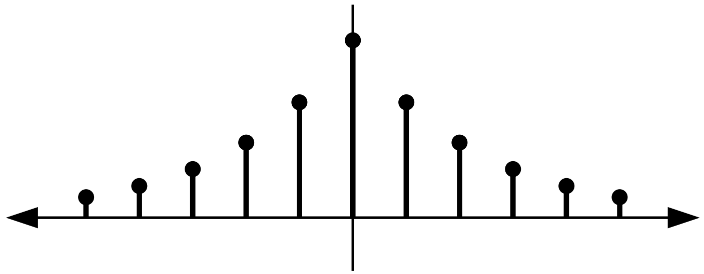
$$x(t) = \sum_{j=-\infty}^{\infty} \xi_j e^{\mathrm{i}\omega_j t} \quad \longleftrightarrow \quad \xi_j = \frac{1}{T}\int_{0}^{T} x(t)e^{-\mathrm{i}\omega_j t}dt$$

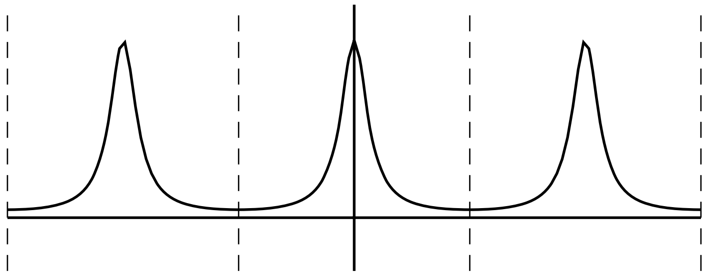




(iii) The discrete-time Fourier transform:

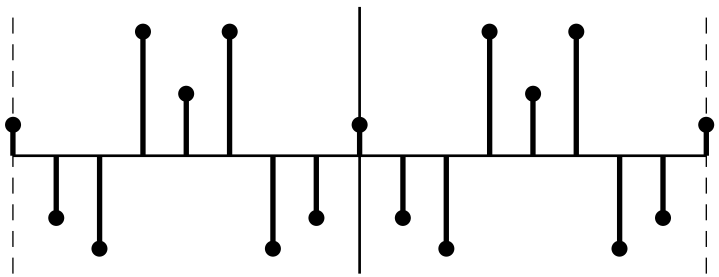
$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega) e^{i\omega t} d\omega \qquad \longleftrightarrow \qquad \xi(\omega) = \sum_{t=-\infty}^{\infty} x_t e^{-i\omega t}$$

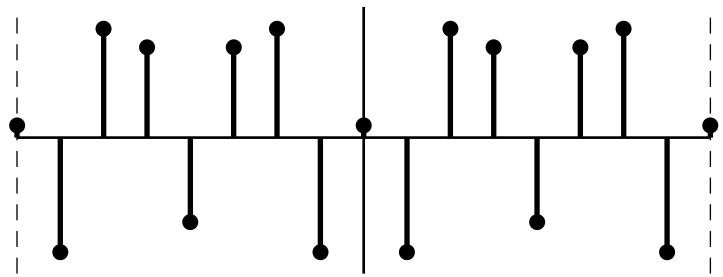




(iv) The discrete Fourier transform:

$$x_t = \sum_{j=0}^{T-1} \xi_j e^{i\omega_j t} \qquad \longleftrightarrow \qquad \xi_j = \frac{1}{T}\sum_{t=0}^{T-1} x_t e^{-i\omega_j t}$$





The relationship between the continuous periodic function and its *Fourier series transform* can be summarised by writing

$$x(t) = \sum_{j=-\infty}^{\infty} \xi_j e^{\mathrm{i}\omega j t} \qquad \longleftrightarrow \qquad \xi_j = \frac{1}{T}\int_0^T x(t)e^{-\mathrm{i}\omega j t}dt. \tag{5.1}$$

The *discrete-time Fourier transform* reverses the role of the time and frequency domains. It maps an absolutely summable sequence $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ into a continuous periodic function:

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega)e^{\mathrm{i}\omega t}d\omega \qquad \longleftrightarrow \qquad \xi(\omega) = \sum_{t=-\infty}^{\infty} x_t e^{-\mathrm{i}\omega t}. \tag{5.2}$$

The *Fourier integral transform* maps an absolutely integrable function from one domain to a similar function in the other domain:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega)e^{\mathrm{i}\omega t}d\omega \longleftrightarrow \xi(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}\omega t}dt, \tag{5.3}$$

whereas a *discrete Fourier transform* maps from a finite sequence $\{x_t; t = 0, 1, 2, \ldots, T-1\}$ in the time domain to a finite sequence $\{\omega_j = 2\pi j/T; j = 0, 1, 2, \ldots, T-1\}$ in the frequency domain:

$$x_t = \sum_{j=0}^{T-1} \xi_j e^{\mathrm{i}\omega_j t} \qquad \longleftrightarrow \qquad \xi_j = \frac{1}{T}\sum_{t=0}^{T-1} x_t e^{-\mathrm{i}\omega_j t}. \tag{5.4}$$

The Classical Fourier Series

The Fourier series expresses a continuous, or piecewise continuous, period function $x(t) = x(t+T)$ in terms of a sum of sine and cosine functions of harmonically increasing frequencies $\{\omega_j = 2\pi j/T; j = 1, 2, 3, \ldots\}$:

$$\begin{aligned} x(t) &= \alpha_0 + \sum_{j=1}^{\infty} \alpha_j \cos(\omega_j t) + \sum_{j=1}^{\infty} \beta_j \sin(\omega_j t) \\ &= \alpha_0 + \sum_{j=1}^{\infty} \rho_j \cos(\omega_j t - \theta_j). \end{aligned} \tag{5.5}$$

Here, $\omega_1 = 2\pi/T$ is the fundamental frequency. The second expression depends upon the definitions

$$\rho_j^2 = \alpha_j^2 + \beta_j^2 \quad \text{and} \quad \theta_j = \tan^{-1}(\beta_j/\alpha_j). \tag{5.6}$$

The equality follows from the trigonometrical identity

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B). \tag{5.7}$$

According to Euler's equations, there are

$$\cos(\omega_j t) = \frac{1}{2}(e^{\mathrm{i}\omega_j t} + e^{-\mathrm{i}\omega_j t}) \quad \text{and} \quad \sin(\omega_j t) = \frac{-\mathrm{i}}{2}(e^{\mathrm{i}\omega_j t} - e^{-\mathrm{i}\omega j t}). \tag{5.8}$$

Therefore, equation (5.5) can be expressed as

$$x(t) = \alpha_0 + \sum_{j=1}^{\infty} \frac{\alpha_j + \mathrm{i}\beta_j}{2} e^{-\mathrm{i}\omega_j t} + \sum_{j=1}^{\infty} \frac{\alpha_j - \mathrm{i}\beta_j}{2} e^{\mathrm{i}\omega_j t}, \tag{5.9}$$

which can be written concisely as

$$x(t) = \sum_{j=-\infty}^{\infty} \xi_j e^{\mathrm{i}\omega_j t}, \tag{5.10}$$

where

$$\xi_0 = \alpha_0, \quad \xi_j = \frac{\alpha_j - \mathrm{i}\beta_j}{2} \quad \text{and} \quad \xi_{-j} = \xi_j^* = \frac{\alpha_j + \mathrm{i}\beta_j}{2}. \tag{5.11}$$

The relationship between the continuous periodic function and its Fourier transform can be summarised by writing

$$x(t) = \sum_{j=-\infty}^{\infty} \xi_j e^{\mathrm{i}\omega_j t} \quad \longleftrightarrow \quad \xi_j = \frac{1}{T}\int_0^T x(t) e^{-\mathrm{i}\omega_j t} dt. \tag{5.12}$$

The Discrete-Time Fourier Transform

The discrete-time Fourier transform transforms square summable a sequence $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ within the time domain, into a continuous periodic function in the frequency domain. It interchanges the two domains of the classical Fourier series transform:

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega)e^{\mathrm{i}\omega t}d\omega \quad \longleftrightarrow \quad \xi(\omega) = \sum_{t=-\infty}^{\infty} x_t e^{-\mathrm{i}\omega t}. \tag{5.13}$$

The periodicity is now in the frequency domain such that $\xi(\omega) = \xi(\omega + 2\pi)$ for all ω. The periodic function completes a single cycle in any interval of length 2π. It may be appropriate to define the function over the interval $[0, 2\pi]$ instead of the interval $[-\pi, \pi]$.

The spectral density function of a stationary process is obtained from discrete-time FT of its autocovariance sequence:

$$f(\omega) = \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-\mathrm{i}\omega\tau} = \frac{1}{2\pi}\left\{\gamma_0 + \sum_{\tau=-\infty}^{\infty} \gamma_\tau \cos(\omega\tau)\right\}. \tag{5.14}$$

Notice that this is a cosine Fourier transform by virtue of the symmetry of the autocovariance function: $\gamma_\tau = \gamma_{-\tau}$. Also, the factor $1/2\pi$ has migrated from one domain to the other.

The Fourier Integral Transform

In common with the discrete Fourier Transform, the integral transform is symmetric with respect to the two domains:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega)e^{\mathrm{i}\omega t}d\omega \quad \longleftrightarrow \quad \xi(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{i}\omega t}dt. \tag{5.15}$$

A sequence that has been sampled at the integer time points from a continuous aperiodic function that is square-integrable will have a periodic transform. Let $\xi(\omega)$ be the transform of the continuous aperiodic function, and let $\xi_S(\omega)$ be the transform of the sampled sequence $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$. Then

$$x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega)e^{\mathrm{i}\omega t}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi_S(\omega)e^{\mathrm{i}\omega t}d\omega. \tag{5.16}$$

The equality of the two integrals implies that

$$\xi_S(\omega) = \sum_{j=-\infty}^{\infty} \xi(\omega + 2j\pi). \tag{5.17}$$

Thus, the periodic function $\xi_S(\omega)$ is obtained by wrapping $\xi(\omega)$ around a circle of circumference of 2π and adding the coincident ordinates.

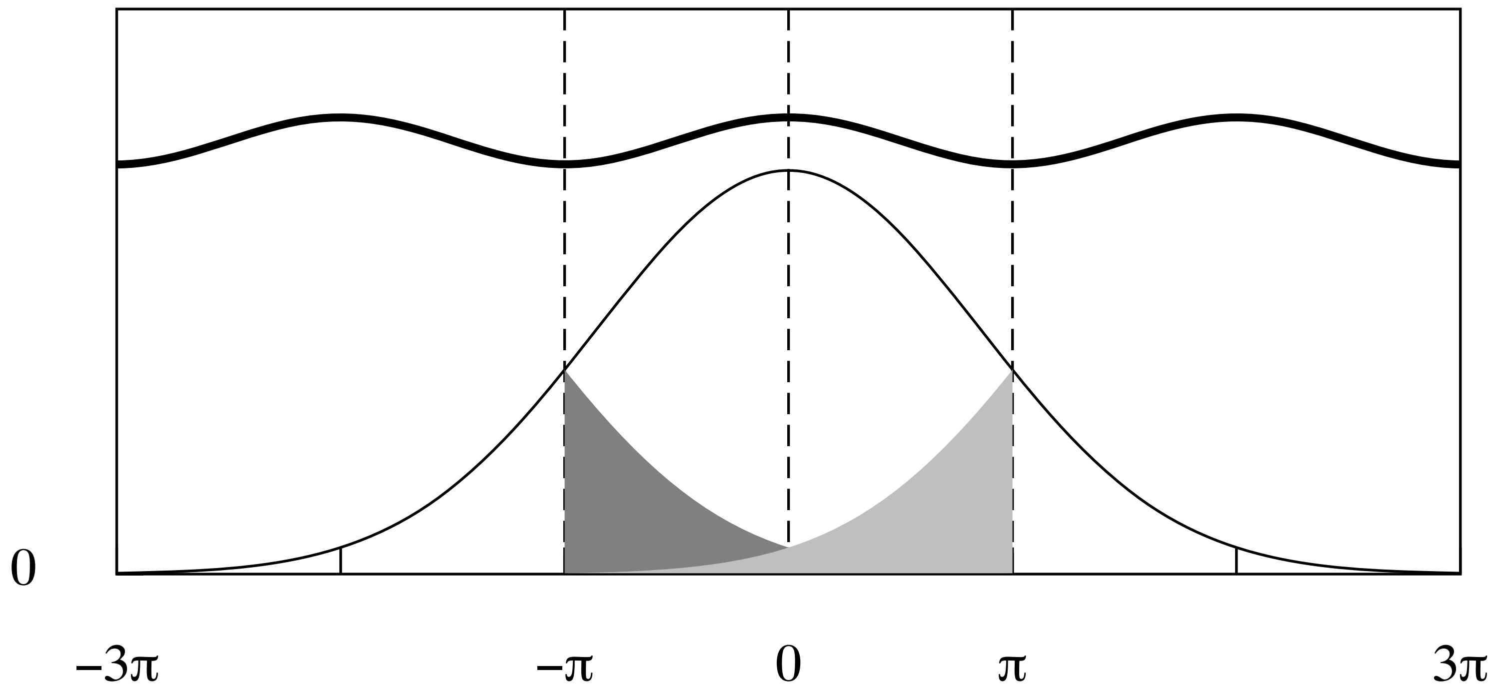


Figure. The figure illustrates the aliasing effect of regular sampling. The bell-shaped function supported on the interval $[-3\pi, 3\pi]$ is the spectrum of a continous-time process. The spectrum of the sampled process, represented by the heavy line, is a periodic function of period 2π.

The effect of sampling is to wrap the spectrum around a circle of radius 2π and to add the overlying parts. The same effect is obtained by folding the branches of the function, supported on $[-3\pi, -\pi]$ and $[\pi, 3\pi]$, onto the interval $[-\pi, \pi]$.

Sampling and Sinc-Function Interpolation

If $\xi(\omega) = \xi_S(\omega)$ is a continuous function band-limited to the interval $[-\pi, \pi]$, then it may be regarded as a periodic function of a period of 2π. Putting

$$\xi_S(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega} \quad \text{into} \quad x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t}\xi_S(\omega)d\omega \tag{5.18}$$

gives

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}k\omega}\right\} e^{\mathrm{i}\omega t}d\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x_k \int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)}d\omega. \end{aligned} \tag{5.19}$$

The integral on the RHS, which gives rise to the so-called sinc function, is evaluated as

$$\int_{-\pi}^{\pi} e^{\mathrm{i}\omega(t-k)}d\omega = 2\frac{\sin\{\pi(t-k)\}}{t-k}. \tag{5.20}$$

Putting this into the RHS gives

$$x(t) = \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} x_k \phi(t-k). \tag{5.21}$$

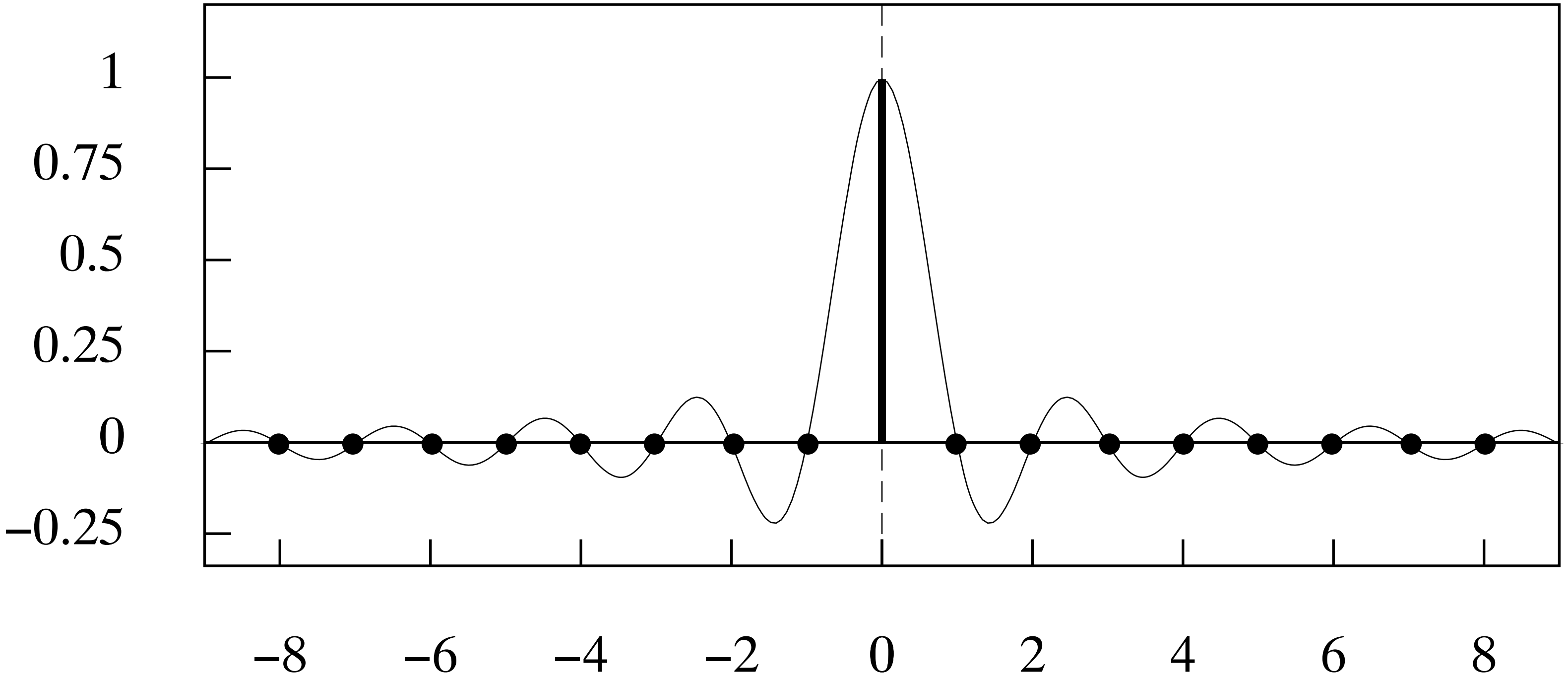


Figure. The sinc function kernel $\phi(t) = \sin(\pi t)/\pi t$.