3. THE DISCRETE FOURIER TRANSFORM

Complex Numbers

There are three ways of representing the conjugate complex numbers λ and λ^*:

$$\begin{aligned} \lambda &= \alpha + \mathrm{i}\beta = \rho(\cos\theta + \mathrm{i}\sin\theta) = \rho e^{\mathrm{i}\theta}, \\ \lambda^* &= \alpha - \mathrm{i}\beta = \rho(\cos\theta - \mathrm{i}\sin\theta) = \rho e^{-\mathrm{i}\theta}. \end{aligned} \tag{3.1}$$

Here, there are

$$\rho = \sqrt{\alpha^2 + \beta^2} \qquad \text{and} \qquad \tan\theta = \beta/\alpha. \tag{3.2}$$

These are the Cartesian form, the trigonometrical form and the exponential form. The parameter $\rho = |\lambda|$ is the modulus of the roots and the parameter $\theta = \arg(\lambda)$, is the argument of the exponential form. The Cartesian and trigonometrical representations are understood by considering the Argand diagram ovrleaf.

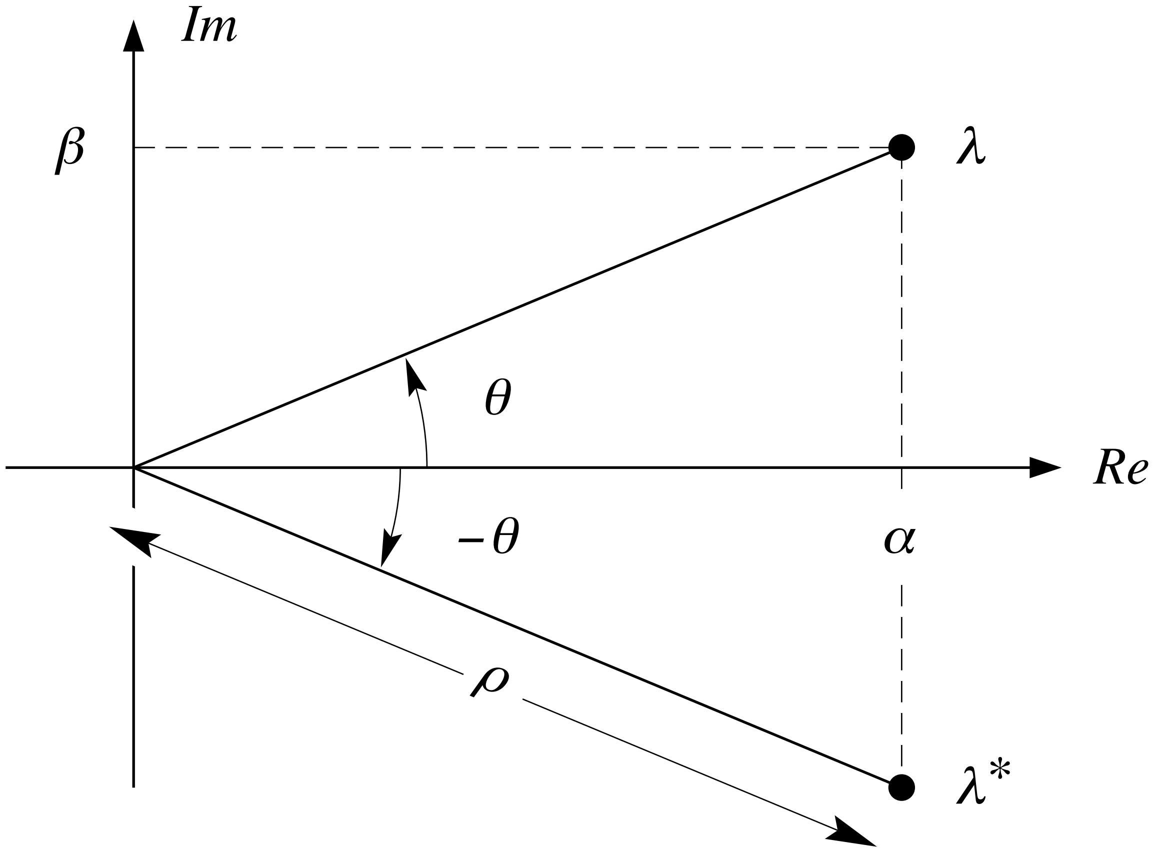

Figure The Argand diagram showing a complex number $\lambda = \alpha + i\beta$ and its conjugate $\lambda^* = \alpha - i\beta$.

The exponential form is understood by considering the series expansions of $\cos\theta$ and $\mathrm{i}\sin\theta$ about the point $\theta = 0$:

$$\begin{aligned}\cos\theta &= \left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right\},\\ \mathrm{i}\sin\theta &= \left\{\mathrm{i}\theta - \frac{\mathrm{i}\theta^3}{3!} + \frac{\mathrm{i}\theta^5}{5!} - \frac{\mathrm{i}\theta^7}{7!} + \cdots\right\}.\end{aligned}\tag{3.3}$$

Adding the series gives

$$\begin{aligned}\cos\theta + \mathrm{i}\sin\theta &= \left\{1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \frac{\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots\right\}\\ &= e^{\mathrm{i}\theta}.\end{aligned}\tag{3.4}$$

Likewise, subtraction gives

$$\cos\theta - \mathrm{i}\sin\theta = e^{-\mathrm{i}\theta}.\tag{3.5}$$

Equations (3.4) and (3.5) are known as Euler's formulae. The inverse formulae are

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2}\tag{3.6}$$

and

$$\sin\theta = -\frac{\mathrm{i}}{2}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}) = \frac{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}}{2\mathrm{i}}.\tag{3.7}$$

Trigonometrical Identities

The addition theorems or compound-angle theorems are familiar from elementary trigonometry where they are proved by geometric means:

$$\begin{aligned}
&\text{(a)} \qquad \cos(A+B) = \cos A\cos B - \sin A\sin B,\\
&\text{(b)} \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B,\\
&\text{(c)} \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B,\\
&\text{(d)} \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B.
\end{aligned} \tag{3.8}$$

We can also prove these using Euler's equations from (3.6) and (3.7). Consider, for example, the first equation (a). We have

$$\begin{aligned}
\cos(A+B) &= \frac{1}{2}\Big\{\exp\big(\mathrm{i}[A+B]\big) + \exp\big(-\mathrm{i}[A+B]\big)\Big\}\\
&= \frac{1}{2}\Big\{\exp(\mathrm{i}A)\exp(\mathrm{i}B) + \exp(-\mathrm{i}A)\exp(-\mathrm{i}B)\Big\}\\
&= \frac{1}{2}\Big\{(\cos A + \mathrm{i}\sin A)(\cos B + \mathrm{i}\sin B) +\\
&\qquad\qquad (\cos A - \mathrm{i}\sin A)(\cos B - \mathrm{i}\sin B)\Big\}\\
&= \cos A\cos B - \sin A\sin B.
\end{aligned} \tag{3.9}$$

The other relationships are established likewise.

From the addition theorems, we can directly establish the following sum–product transformations:

$$
\begin{aligned}
&\text{(a)} \qquad \sin A \cos B = \frac{1}{2}\big\{\sin(A+B) + \sin(A-B)\big\},\\
&\text{(b)} \qquad \cos A \sin B = \frac{1}{2}\big\{\sin(A+B) - \sin(A-B)\big\},\\
&\text{(c)} \qquad \cos A \cos B = \frac{1}{2}\big\{\cos(A+B) + \cos(A-B)\big\},\\
&\text{(d)} \qquad \sin A \sin B = \frac{1}{2}\big\{\cos(A-B) - \cos(A+B)\big\}.
\end{aligned}
\tag{3.10}
$$

Trigonometrical Orthogonality Conditions

$$\text{(a)}\qquad \int_0^{2\pi}\cos(jx)\cos(kx)dx = \begin{cases} 0, & \text{if } j \neq k; \\ \pi, & \text{if } j = k > 0; \\ 2\pi, & \text{if } j = k = 0; \end{cases}$$

$$\text{(b)}\qquad \int_0^{2\pi}\sin(jx)\sin(kx)dx = \begin{cases} 0, & \text{if } j \neq k; \\ \pi, & \text{if } j = k > 0; \end{cases} \tag{3.11}$$

$$\text{(c)}\qquad \int_0^{2\pi}\cos(jx)\sin(kx)dx = 0, \quad \text{for all} \quad j, k.$$

To prove the results in (3.11)(a), we may use (3.10)(c) to rewrite the integral as

$$\int_0^{2\pi}\cos(jx)\cos(kx)dx = \frac{1}{2}\int_0^{2\pi}\Big\{\cos([j+k]x) + \cos([j-k]x)\Big\}dx.$$

If $j \neq k$, then both the cosine terms complete an integral number of cycles over the range $[0, 2\pi]$; and, therefore, they integrate to zero. If $j = k > 0$, then the second cosine becomes unity, and, therefore, it integrates to 2π over the range $[0, 2\pi]$ whilst the first cosine term integrates to zero. If $j = k = 0$, then both cosine terms become unity and both have integrals of 2π.

The Fourier Decomposition of a Time Series

An arbitrary sequence $\{y_t; t = 0, 1, \ldots, T-1\}$ of $T = 2n$ data points can expressed in terms of T sinusoidal functions. If

$$\omega_j = \frac{2\pi j}{T}, \quad j = 0, \ldots, n = \frac{T}{2}, \tag{3.12}$$

which are at equally spaced points in the interval $[0, \pi]$, then

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{3.13}$$

Let $c_j = [c_{0j}, \ldots, c_{T-1,j}]'$ and $s_j = [s_{0,j}, \ldots, s_{T-1,j}]'$ represent vectors of T values of the functions $\cos(\omega_j t)$ and $\sin(\omega_j t)$, respectively. Then, there are the following orthogonality conditions:

$$\begin{aligned} c_i'c_j &= 0 \quad \text{if} \quad i \neq j, \\ s_i's_j &= 0 \quad \text{if} \quad i \neq j, \\ c_i's_j &= 0 \quad \text{for all} \quad i, j. \end{aligned} \tag{3.14}$$

In addition, there are the following sums of squares:

$$\begin{aligned} c_0'c_0 &= c_n'c_n = T, \\ s_0's_0 &= s_n's_n = 0, \\ c_j'c_j &= s_j's_j = \frac{T}{2}. \end{aligned} \tag{3.15}$$

The vector $c_0 = [1, 1, \ldots, 1]' = \iota$ is just the summation vector. The 'regression' formulae for the Fourier coefficients are therefore

$$\alpha_0 = (\iota'\iota)^{-1}\iota' y = \frac{1}{T}\sum_t y_t = \bar{y}, \tag{3.16}$$

$$\alpha_j = (c_j' c_j)^{-1} c_j' y = \frac{2}{T}\sum_t y_t \cos\omega_i t, \tag{3.17}$$

$$\beta_j = (s_j' s_j)^{-1} s_j' y = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{3.18}$$

$$\alpha_n = (c_n' c_n)^{-1} c_n' y = \frac{1}{T}\sum_t (-1)^t y_t, \quad \text{if} \quad T = 2n. \tag{3.19}$$

The Periodogram and the Spectral Analysis of Variance

There is a complete decomposition of the sum of squares of the elements of y which is analogous to that of a regression analysis:

$$y'y = \alpha_0^2 \iota'\iota + \sum_j \alpha_j^2 c_j' c_j + \sum_j \beta_j^2 s_j' s_j. \tag{3.20}$$

Consider writing $\alpha_0^2 \iota'\iota = \bar{y}^2 \iota'\iota = \bar{y}'\bar{y}$ where $\bar{y}' = [\bar{y}, \ldots, \bar{y}]$ is the vector whose repeated element is the sample mean $\bar{y}$. It follows that $y'y - \alpha_0^2 \iota'\iota = y'y - \bar{y}'\bar{y} = (y - \bar{y})'(y - \bar{y})$. Therefore, we can rewrite the equation as

$$(y - \bar{y})'(y - \bar{y}) = \frac{T}{2} \sum_j \left\{ \alpha_j^2 + \beta_j^2 \right\} = \frac{T}{2} \sum_j \rho_j^2, \tag{3.21}$$

and it follows that we can express the variance of the sample as

$$\begin{aligned} \frac{1}{T} \sum_{t=0}^{T-1} (y_t - \bar{y})^2 &= \frac{1}{2} \sum_{j=1}^{n} (\alpha_j^2 + \beta_j^2) \\ &= \frac{2}{T^2} \sum_j \left\{ \left(\sum_t y_t \cos \omega_j t \right)^2 + \left(\sum_t y_t \sin \omega_j t \right)^2 \right\}. \end{aligned} \tag{3.22}$$

The proportion of the variance which is attributable to the component at frequency ω_j is $(\alpha_j^2 + \beta_j^2)/2 = \rho_j^2/2$, where ρ_j is the amplitude of the component.

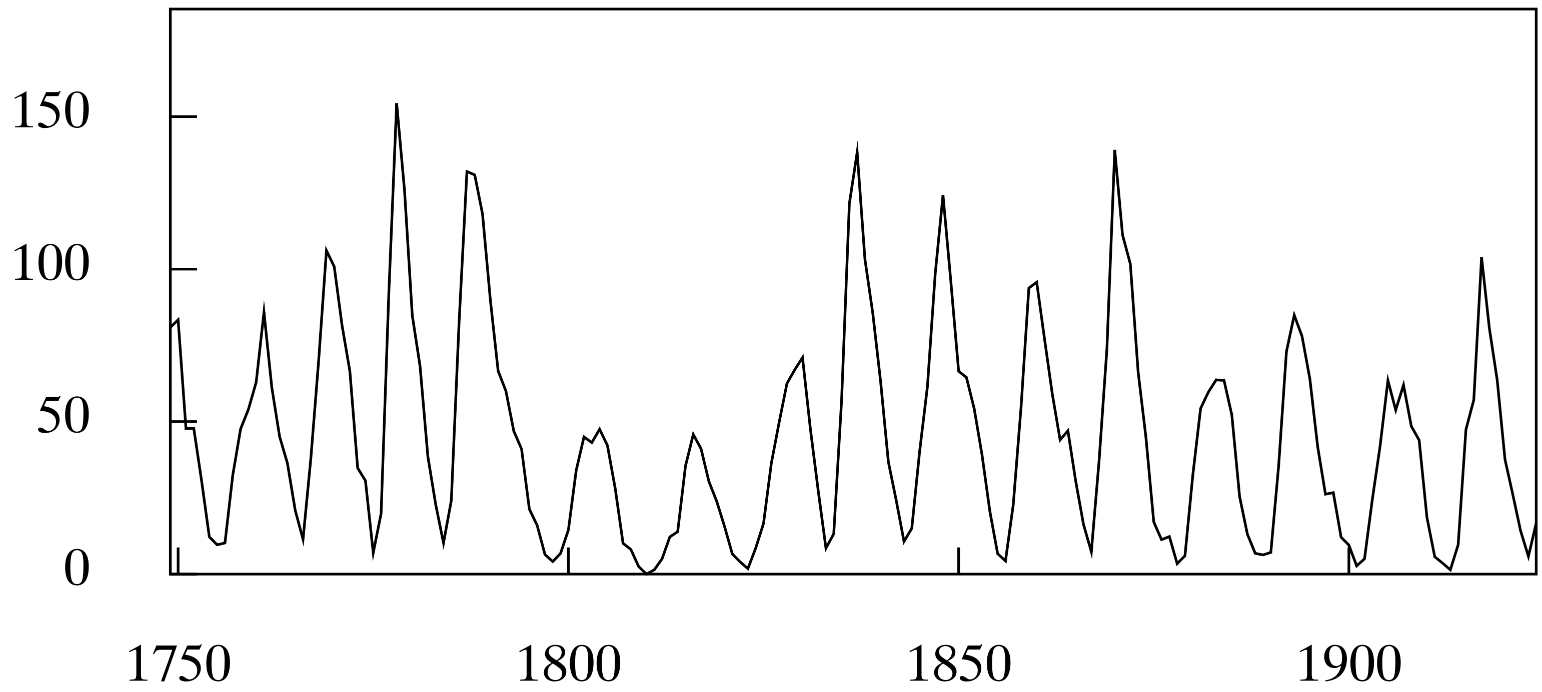

Figure. Wolfer's Sunspot numbers 1749–1924

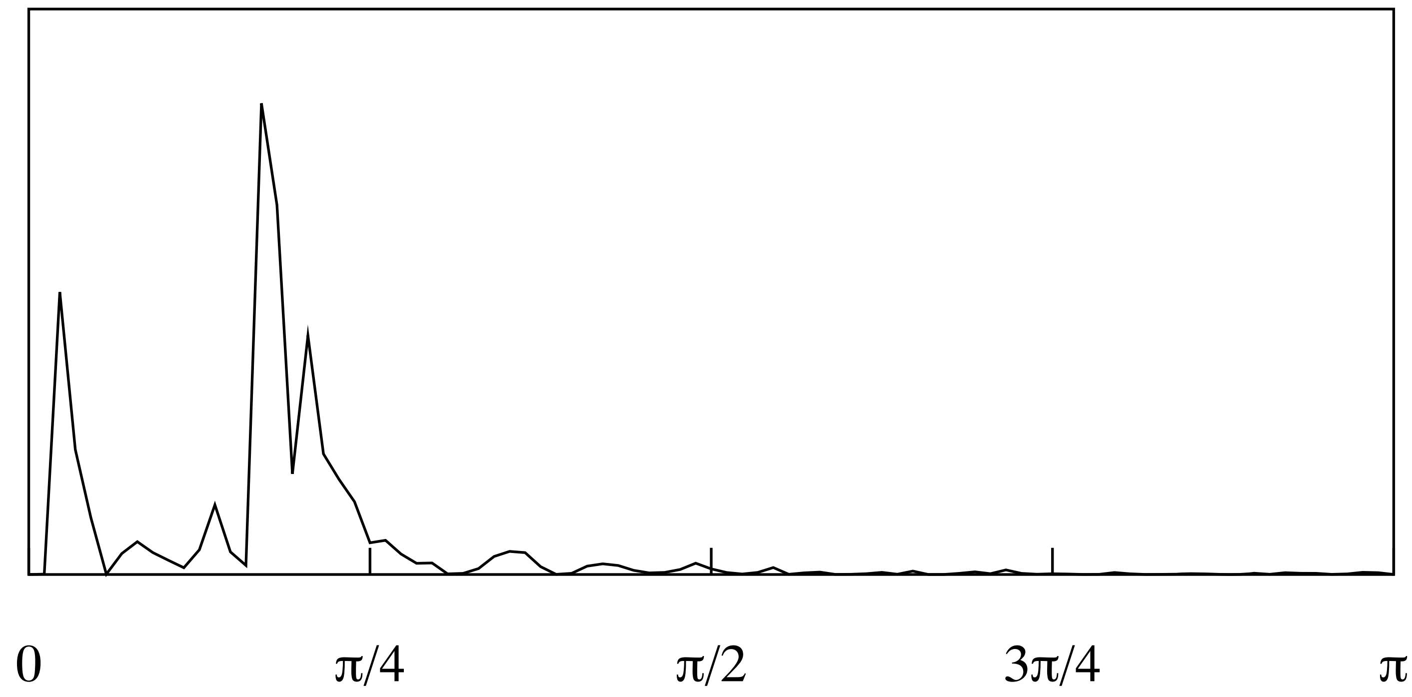

Figure 3. The periodogram of Wolfer's Sunspot numbers 1749–1924.

The Periodogram and the Empirical Autocovaraiances

A natural way of representing the serial dependence of the elements of a data sequence is to estimate their autocovariances. The empirical autocovariance of lag τ is

$$c_\tau = \frac{1}{T}\sum_{t=\tau}^{T-1}(y_t - \bar{y})(y_{t-\tau} - \bar{y}). \tag{3.23}$$

The empirical autocorrelation of lag τ, defined by $r_\tau = c_\tau/c_0$, provides a measure of the relatedness that is independent of the units of measurement.

It is straightforward to establish the relationship between the periodogram and the sequence of autocovariances. The periodogram may be written as

$$I(\omega_j) = \frac{2}{T}\left[\left\{\sum_{t=0}^{T-1}\cos(\omega_j t)(y_t - \bar{y})\right\}^2 + \left\{\sum_{t=0}^{T-1}\sin(\omega_j t)(y_t - \bar{y})\right\}^2\right]. \tag{3.24}$$

The identity $\sum_t \cos(\omega_j t)(y_t - \bar{y}) = \sum_t \cos(\omega_j t)y_t$ follows from the fact that, by construction, $\sum_t \cos(\omega_j t) = 0$ for all j.

Expanding the squares in (3.24) gives

$$\begin{aligned} I(\omega_j) =& \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j t)\cos(\omega_j s)(y_t-\bar{y})(y_s-\bar{y})\right\} \\ &+\frac{2}{T}\left\{\sum_t\sum_s \sin(\omega_j t)\sin(\omega_j s)(y_t-\bar{y})(y_s-\bar{y})\right\}, \end{aligned} \tag{3.25}$$

and, via the identity $\cos(A)\cos(B)+\sin(A)\sin(B)=\cos(A-B)$, this becomes

$$I(\omega_j) = \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j[t-s])(y_t-\bar{y})(y_s-\bar{y})\right\}. \tag{3.26}$$

Next, on defining $\tau = t-s$ and writing $c_\tau = \sum_t(y_t-\bar{y})(y_{t-\tau}-\bar{y})/T$, the expression is reduced to

$$I(\omega_j) = 2\sum_{\tau=1-T}^{T-1}\cos(\omega_j\tau)c_\tau, \tag{3.27}$$

which is a Fourier transform of the sequence of empirical autocovariances.

Complex Exponential Forms

The equation

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{3.28}$$

can be written in a more concisely using

$$\cos(\omega_j t) = \frac{1}{2}(e^{\mathrm{i}\omega_j t} + e^{-\mathrm{i}\omega_j t}) \quad \text{and} \quad \sin(\omega_j t) = \frac{-\mathrm{i}}{2}(e^{\mathrm{i}\omega_j t} - e^{-\mathrm{i}\omega_j t}). \tag{3.29}$$

Then

$$\begin{aligned} y_t &= \sum_{j=0}^{n} \Big(\frac{\alpha_j - \mathrm{i}\beta_j}{2} \Big) e^{\mathrm{i}\omega_j t} + \sum_{j=0}^{n} \Big(\frac{\alpha_j + \mathrm{i}\beta_j}{2} \Big) e^{-\mathrm{i}\omega_j t} \\ &= \sum_{j=0}^{n} \zeta_j e^{\mathrm{i}\omega_j t} + \sum_{j=0}^{n} \zeta_j^* e^{-\mathrm{i}\omega_j t} = \sum_{j=-n}^{n} \zeta_j e^{\mathrm{i}\omega_j t}, \end{aligned} \tag{3.30}$$

where $\zeta_j = (\alpha_j - \mathrm{i}\beta_j)/2$, which has $\zeta_j^* = \zeta_{-j} = (\alpha_j + \mathrm{i}\beta_j)/2$ as its complex conjugate. Also $\zeta_0 = \alpha_0$ and $\zeta_n = \alpha_n$.

The exponential $\exp(-\mathrm{i}\omega_j) = \exp(-\mathrm{i}2\pi j/T)$ is T-periodic in the index j. Therefore, $\exp(-\mathrm{i}\omega_j) = \exp(-\mathrm{i}\omega_{T-j})$. Likewise for $\exp(\mathrm{i}\omega_j)$. By taking $\zeta_j^* = \zeta_{-j} = \zeta_{T-j}$, we may write

$$y_t = \sum_{j=0}^{T-1} \zeta_j e^{\mathrm{i}\omega_j t}, \tag{3.31}$$

wherein the time and frequency indices are $t, j = 0, 1, \ldots, T-1$. The sequence $\zeta_0, \zeta_1, \ldots, \zeta_{T-1}$ constitute the spectral ordinates of the data. The inverse of (3.31) is the transform that maps from the data to the spectral ordinates:

$$\zeta_j = \frac{1}{T} \sum_{t=0}^{T-1} y_t e^{-\mathrm{i}\omega_j t}. \tag{3.32}$$

Equations (3.31) and (3.32) summarise the discrete Fourier transform.