2. PROPERTIES OF TRANSFER FUNCTIONS

The Impulse Response function

A transfer function can be characterised by its effect on certain elementary reference signals. The simplest of these is the impulse sequence defined by

$$\delta_t = \begin{cases} 1, & \text{if } t = 0; \\ 0, & \text{if } t \neq 0. \end{cases} \tag{2.1}$$

Its z-transform is $\delta(z) = 1$. The output engendered by the impulse is described as the impulse response function. The impulse response of the rational function $\beta(z)/\alpha(z)$ is the sequence $\{\omega_0, \omega_1, \ldots\}$ of the coefficients of its series expansion

Another common reference signal is the unit step defined by

$$s_t = \begin{cases} 1, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0. \end{cases} \tag{2.2}$$

The step response is the sequence of the partial sums of the impulse response. The sequence converges to

$$\gamma = \omega(1) = \frac{\beta(1)}{\alpha(1)} = \frac{\beta_0 + \beta_1 + \cdots + \beta_q}{\alpha_0 + \alpha_1 + \cdots + \alpha_p}, \tag{2.3}$$

which is described as the steady-state gain of the transfer function.

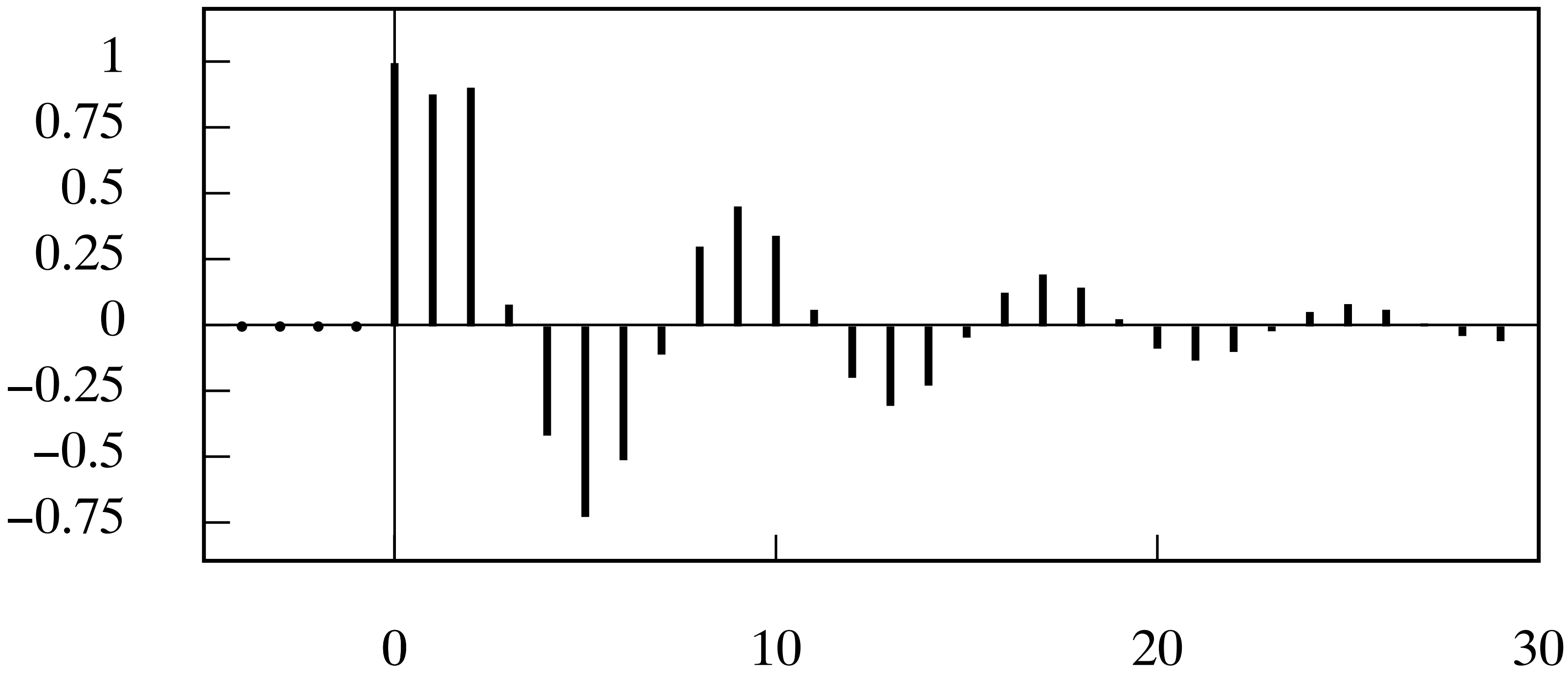

Figure 1. The impulse response of the transfer function $b(z)/a(z)$ with $a(z) = 1.0 - 0.673z + 0.463z^2 + 0.486z^3$ and $b(z) = 1.0 + 0.208z + 0.360z^2$.

Stability of a Transfer Function

The stability of a rational transfer function $\beta(z)/\alpha(z)$ can be investigated using its partial fraction decomposition. Assume that $\beta(z)/\alpha(z)$ is a proper rational function in which the denominator is factorised as

$$\alpha(z) = \prod_{j=1}^{r}(1 - z/\lambda_j)^{n_j}, \tag{2.4}$$

where n_j is the multiplicity of the root λ_j, and $\sum_j n_j = p$ is the degree of the polynomial. Then,

$$\frac{\beta(z)}{\alpha(z)} = \sum_{j=1}^{r}\sum_{k=1}^{n_j}\frac{c_{j,k}}{(1 - z/\lambda_j)^k}; \tag{2.5}$$

and the task is to find the series expansions of the partial fractions.

Consider a partial fraction that contains a distinct (unrepeated) real root λ. The expansion is

$$\frac{c}{1 - z/\lambda} = c\{1 + z/\lambda + (z/\lambda)^2 + \cdots\}. \tag{2.6}$$

For this to converge for all $|z| \leq 1$, it is necessary and sufficient that $|\lambda| > 1$; and this is necessary and sufficient for the satisfaction of the bounded-input bounded-output (BIBO) stability condition.

When $\alpha(z)$ has conjugate complex roots λ and λ^*, the partial fraction has a quadratic denominator:

$$\frac{gz+d}{(1-z/\lambda)(1-z/\lambda^*)}=\frac{c}{(1-z/\lambda)}+\frac{c^*}{(1-z/\lambda^*)}. \tag{2.7}$$

Then $c=(g\lambda+d)/(1-\lambda/\lambda^*)$ and $c^*=(g\lambda^*+d)/(1-\lambda^*/\lambda)$ are also conjugate complex numbers. The expansion of (2.6) applies to complex roots as well as to real roots:

$$\begin{aligned}\frac{c}{1-z/\lambda}+\frac{c^*}{1-z/\lambda^*}&=c\{1+z/\lambda+(z/\lambda)^2+\cdots\}+c^*\{1+z/\lambda^*+(z/\lambda^*)^2+\cdots\}\\&=\sum_{t=0}^{\infty}z^t(c\lambda^{-t}+c^*\lambda^{*-t}).\end{aligned} \tag{2.8}$$

The complex quantities can be represented in terms of exponentials:

$$\lambda=\kappa^{-1}e^{-\mathrm{i}\omega},\quad c=\rho e^{-\mathrm{i}\theta},\qquad \lambda^*=\kappa^{-1}e^{\mathrm{i}\omega},\quad c^*=\rho e^{\mathrm{i}\theta}. \tag{2.9}$$

Then, the generic term in the expansion becomes

$$\begin{aligned}z^t(c\lambda^{-t}+c^*\lambda^{*-t})&=z^t\rho\kappa^t\{e^{\mathrm{i}(\omega t-\theta)}+e^{-\mathrm{i}(\omega t-\theta)}\}\\&=z^t2\rho\kappa^t\cos(\omega t-\theta).\end{aligned} \tag{2.10}$$

The expansion converges for all $|z| \leq 1$ if and only if $|\kappa| < 1$, which is a condition on the complex modulus of κ. But, $|\kappa| = |\lambda^{-1}| = |\lambda|^{-1}$; so it is confirmed that the necessary and sufficient condition for convergence is that $|\lambda| > 1$.

In the case of a repeated root with a multiplicity of n, a binomial expansion is available that gives

$$\frac{1}{(1 - z/\lambda)^n} = 1 + n\frac{z}{\lambda} + \frac{n(n+1)}{2!}\left(\frac{z}{\lambda}\right)^2 + \frac{n(n+1)(n+2)}{3!}\left(\frac{z}{\lambda}\right)^3 + \cdots. \tag{2.11}$$

If λ is real, then $|\lambda| > 1$ is the condition for convergence. If λ is complex, then it can be combined with the conjugate roots in the manner of (2.10) to create a trigonometric function and, again, the condition for convergence is that $|\lambda| > 1$.

The general conclusion is that the transfer function is stable if and only if all of the roots of the denominator polynomial $a(z)$, which are described as the poles of the transfer function, lie outside the unit circle in the complex plane.

It helps to represent the poles (denominator roots) and zeros (numerator roots) of the transfer function graphically by showing their locations within the complex plane. It is more convenient to represent the poles and zeros of $\beta(z^{-1})/\alpha(z^{-1})$, which are the reciprocals of those of $\beta(z)/\alpha(z)$, since, for a stable and invertible transfer function, these must lie within the unit circle.

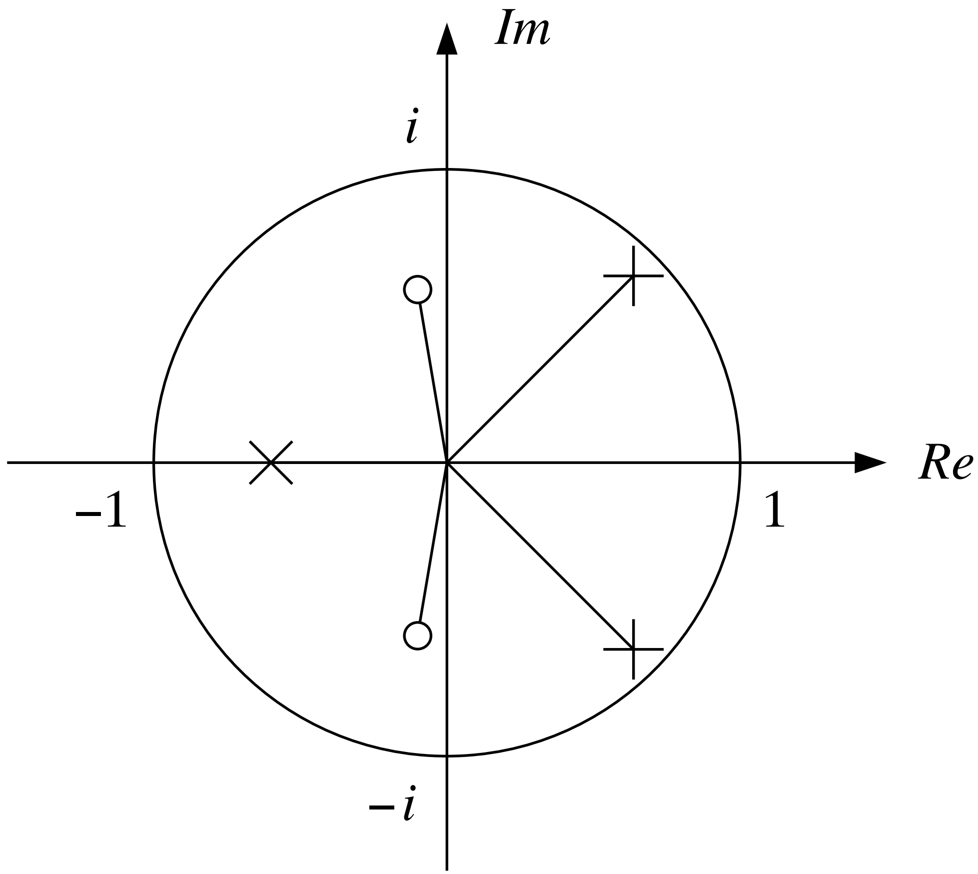

Figure 2. The pole–zero diagram for the transfer function $b(z^{-1})/a(z^{-1})$ corresponding to the impulse response function of Figure 1. The poles are marked by crosses and the zeros by circles.

The Response to a Sinusoidal Input

Consider mapping the signal sequence $\{x_t = \cos(\omega t); t = 0, \pm 1, \pm 2, \ldots\}$ through the transfer function with the coefficients $\psi_0, \psi_1, \ldots, \psi_q$. The output is

$$y(t) = \sum_{j=0}^{q} \psi_j \cos\big(\omega[t-j]\big). \tag{2.12}$$

The trigonometrical identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$ enables us to write this as

$$\begin{aligned} y(t) &= \Big\{\sum_j \psi_j \cos(\omega j)\Big\} \cos(\omega t) + \Big\{\sum_j \psi_j \sin(\omega j)\Big\} \sin(\omega t) \\ &= \alpha \cos(\omega t) + \beta \sin(\omega t) = \rho \cos(\omega t - \theta). \end{aligned} \tag{2.13}$$

Here we have defined

$$\alpha = \sum_{j=0}^{q} \psi_j \cos(\omega j), \quad \beta = \sum_{j=0}^{q} \psi_j \sin(\omega j), \qquad \text{and} \qquad \rho = \sqrt{\alpha^2 + \beta^2}, \quad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right). \tag{2.14}$$

There is a *gain effect* whereby the amplitude of the sinusoid is increased or diminished by a factor of ρ and there is a *phase effect* whereby the peak of the sinusoid is displaced by a time delay of θ/ω periods. The frequency of the output is the same as the frequency of the input.

Aliasing of Sampled Processes

In a discrete-time system, there is a problem of aliasing whereby frequencies (i.e. angular velocities) in excess of π radians per sampling interval are confounded with frequencies within the interval $[0, \pi]$.

Consider the case of a pure cosine wave of unit amplitude and zero phase whose frequency ω lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$\begin{aligned}\cos(\omega t) &= \cos\left\{(2\pi - \omega^*)t\right\} \\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t) \\ &= \cos(\omega^* t);\end{aligned} \tag{2.15}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

Imagine that a person observes the sea level at 6am. and 6pm. each day. He should notice a very gradual recession and advance of the water level; the frequency of the cycle being $f = 1/28$ which amounts to one tide in 14 days. In fact, the true frequency is $f = 1 - 1/28$ which gives 27 tides in 14 days. Observing the sea level every six hours should enable him to infer the correct frequency.

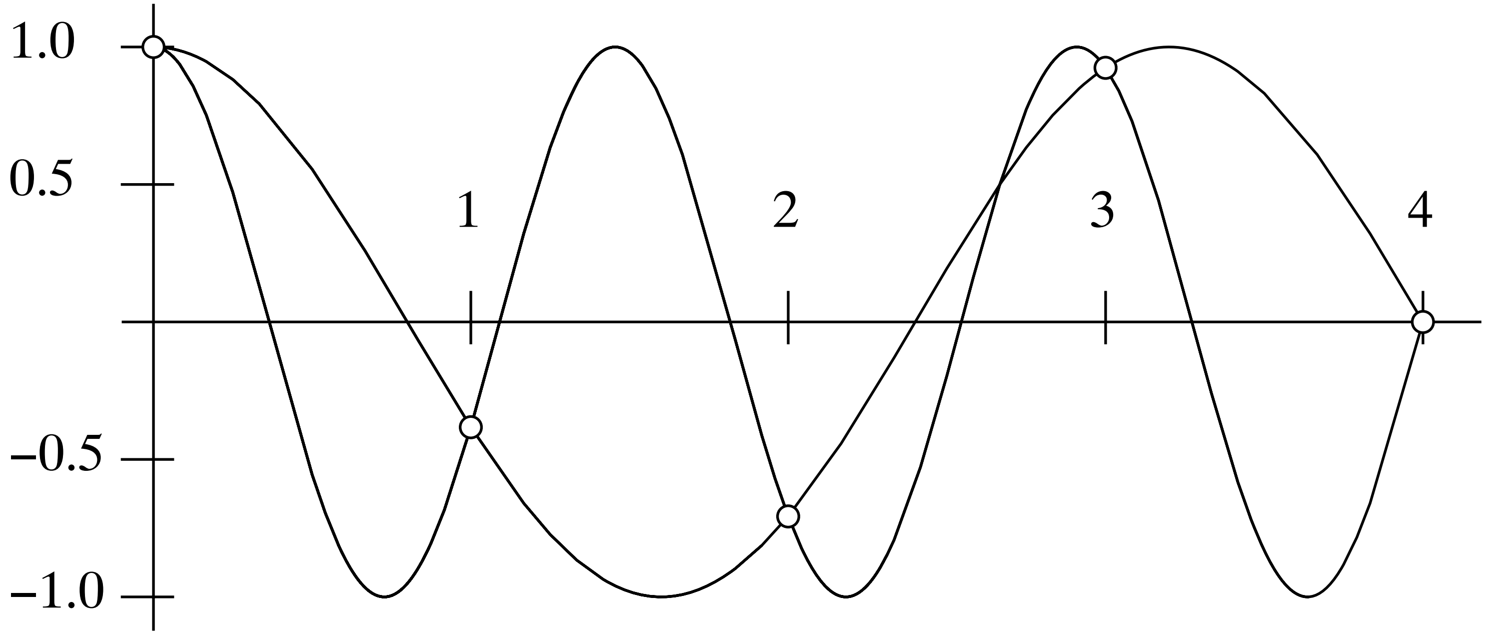

Figure 5. The values of the function $\cos\{(11/8)\pi t\}$ coincide with those of its alias $\cos\{(5/8)\pi t\}$ at the integer points $\{t = 0, \pm 1, \pm 2, \ldots\}$.

Spectral Representation of a Stationary Stochastic Process

Any stationary stochastic process can be expressed as a weighted combination of the infinity of sines and cosines of which the frequencies lie in Nyquist frequency interval $[0, \pi]$. Thus, if x_t is an element of such a process, then it can be represented by

$$x_t = \int_0^\pi \Big\{ \cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega) \Big\}. \tag{2.16}$$

Here, $dA(\omega)$ and $dB(\omega)$ are the infinitesimal increments of stochastic functions defined on the frequency interval that are everywhere continuous but nowhere differentiable. It is assumed that the increments $A(\omega)$ and $B(\omega)$ are uncorrelated with each other and with preceding and succeeding increments.

Such processes can be representing the in terms of complex exponentials. Thus

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{-\mathrm{i}}{2}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}) = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}). \tag{2.17}$$

These enable equation (2.16) to be written as

$$\begin{aligned} x_t &= \int_0^\pi \left\{ \frac{(e^{\mathrm{i}\omega t} + e^{-\mathrm{i}\omega t})}{2} dA(\omega) - \mathrm{i}\frac{(e^{\mathrm{i}\omega t} - e^{-\mathrm{i}\omega t})}{2} dB(\omega) \right\} \\ &= \int_0^\pi \left\{ e^{\mathrm{i}\omega t}\frac{\{dA(\omega) - \mathrm{i}dB(\omega)\}}{2} + e^{-\mathrm{i}\omega t}\frac{\{dA(\omega) + \mathrm{i}dB(\omega)\}}{2} \right\}. \end{aligned} \tag{2.18}$$

On defining

$$dZ(\omega) = \frac{dA(\omega) - \mathrm{i}dB(\omega)}{2} \quad \text{and} \quad dZ^*(\omega) = \frac{dA(\omega) + \mathrm{i}dB(\omega)}{2}$$

and extending the integral over the range $[-\pi, \pi]$, this becomes

$$x_t = \int_{-\pi}^{\pi} e^{\mathrm{i}\omega t} dZ(\omega), \tag{2.19}$$

which is commonly described as the spectral representation of the process generating y_t.

The spectral density function of the process is the function $f(\omega)$ that is defined by

$$E\{dZ(\omega)dZ^*(\omega)\} = E\{dA^2(\omega) + dB^2(\omega)\} = f(\omega)d\omega. \tag{2.20}$$

The increment $f(\omega)d\omega$ is the power or the variance of the process that is attributable to the elements in the frequency interval $[\omega, \omega + d\omega]$; and the integral of $f(\omega)$ over the frequency range $[-\pi, \pi]$ is the overall variance of the process. A white noise process $\{\varepsilon_t\}$ with a variance of σ_ε^2 has a uniform spectral density function $f_\varepsilon(\omega) = \sigma_\varepsilon^2/(2\pi)$.

The Frequency Response

Let $\{\psi_0, \psi_1, \ldots\}$ be the impulse response of the transfer function. Then, the effects of the transfer function upon the spectral elements of the process defined by (2.19) are shown by the equation

$$\begin{aligned} y_t = \sum_j \mu_j x(t-j) &= \sum_j \mu_j \left\{ \int_\omega e^{\mathrm{i}\omega(t-j)} dZ_x(\omega) \right\} \\ &= \int_\omega e^{\mathrm{i}\omega t} \left(\sum_j \mu_j e^{-\mathrm{i}\omega j} \right) dZ_x(\omega). \end{aligned} \tag{2.21}$$

The effects are summarised by the complex-valued frequency-response function

$$\mu(\omega) = \sum \mu_j e^{-\mathrm{i}\omega j} = |\mu(\omega)| e^{-\mathrm{i}\theta(\omega)}. \tag{2.22}$$

The final expression entails the filter gain $|\mu(\omega)|$ and the phase response $\theta(\omega)$:

$$\begin{aligned} |\mu(\omega)|^2 &= \left\{ \sum_{j=0}^{\infty} \mu_j \cos(\omega j) \right\}^2 + \left\{ \sum_{j=0}^{\infty} \mu_j \sin(\omega j) \right\}^2 \\ \theta(\omega) &= \arctan \left\{ \frac{\sum \mu_j \sin(\omega j)}{\sum \mu_j \cos(\omega j)} \right\}. \end{aligned} \tag{2.23}$$

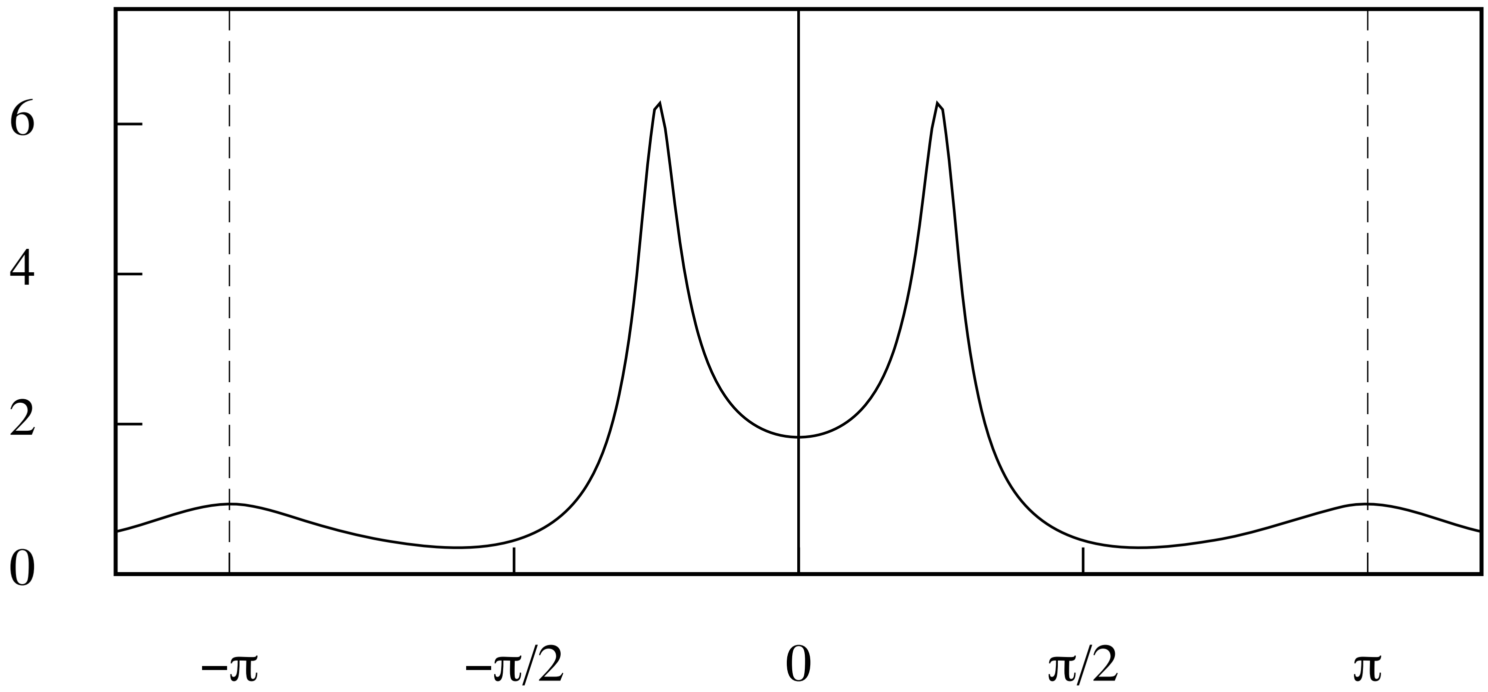

Figure 3. The gain of the transfer function depicted in Figures 1 and 2.

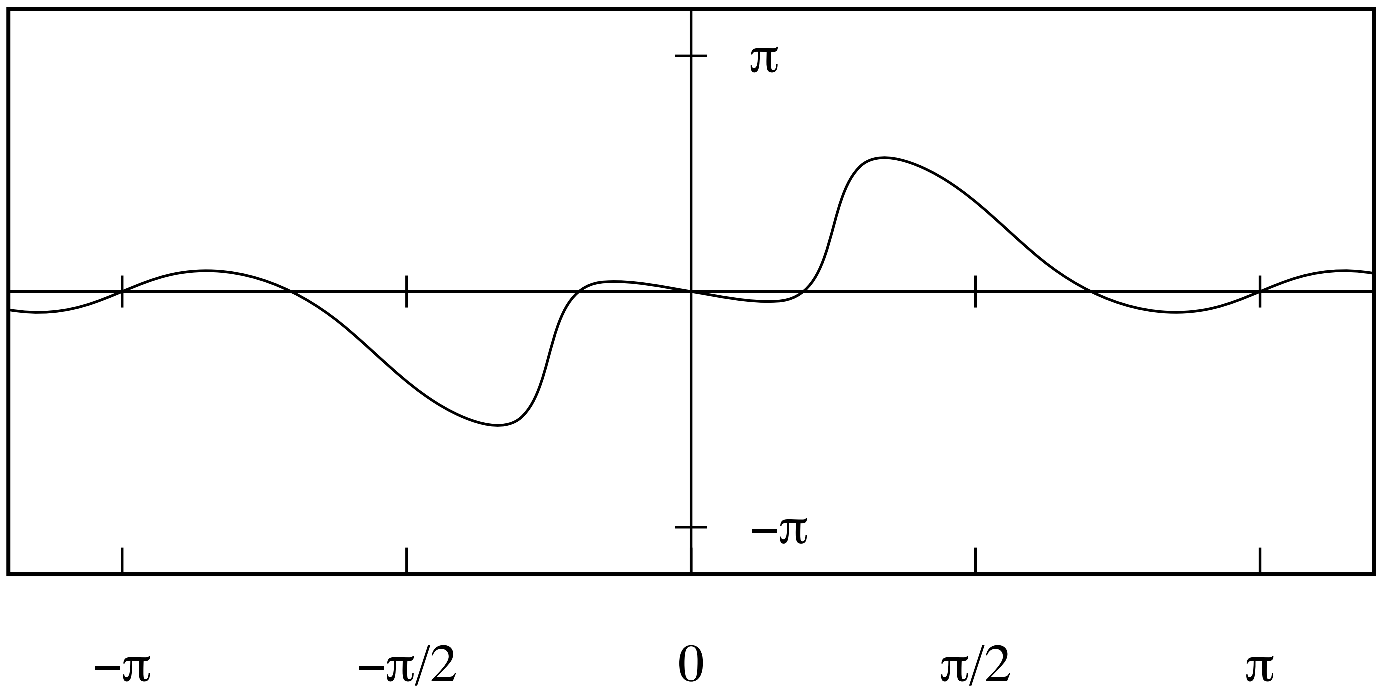

Figure 4. The phase plot to accompany Figure 3.

The frequency response is just the discrete-time Fourier transform of the impulse response function. Equally, it is the z-transform $\psi(z^{-1}) = \sum_j \psi_j z^{-1}$ evaluated at the points $e^{i\omega}$ that lie on the unit circle in the complex plane.

As ω progresses from $-\pi$ to π, or, equally, as $z = e^{i\omega}$ travels around the unit circle, the frequency-response function defines a trajectory in the complex plane, which becomes a closed contour when ω reaches π. The points on the trajectory are characterised by their polar co-ordinates. These are the modulus $|\psi(\omega)|$, which is the length of the radius vector joining $\psi(\omega)$ to the origin, and the argument $\mathrm{Arg}\{\psi(\omega)\} = -\theta(\omega)$ which is the (anticlockwise) angle in radians which the radius makes with the positive real axis.

The spectral density of the output function $f_y(\omega)$ of the filtered process $y(t)$ is given by

$$\begin{aligned} f_y(\omega)d\omega &= E\{dZ_y(\omega)dZ_y^*(\omega)\} \\ &= \mu(\omega)\mu^*(\omega)E\{dZ_x(\omega)dZ_x^*(\omega)\} \\ &= |\mu(\omega)|^2 f_x(\omega)d\omega. \end{aligned} \tag{2.24}$$

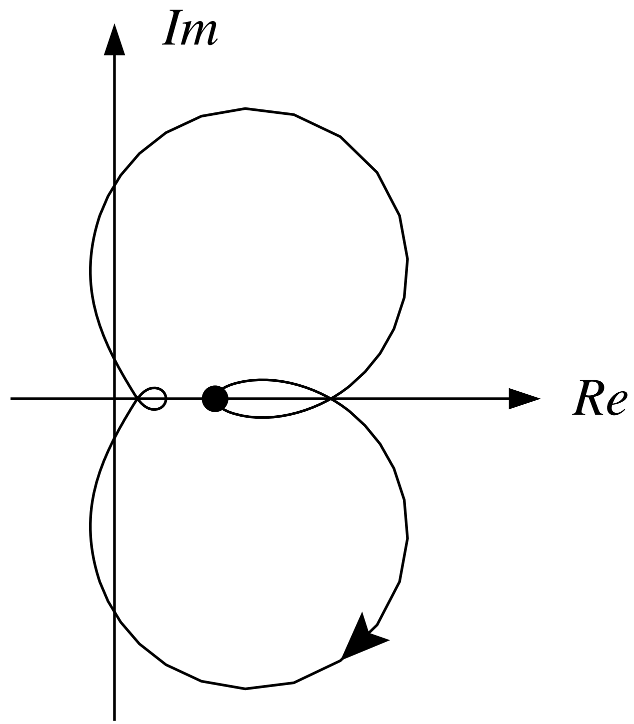

Figure 5. The path described in the complex plane by the frequency response function corresponding to the gain and phase functions of Figures 3 and 4. The trajectory originates, when $\omega = 0$, in the point on the real axis marked by a dot and it travels in the direction of the arrow, of which the tip is reached when $\omega = \pi/4$.