1. THE ALGEBRA FOR TIME-SERIES ANALYSIS

Rational Functions and the z-transform

A univariate and strictly causal transfer function mapping from the input sequence $\{x_t\}$ to the output sequence $\{y_t\}$ can be represented by the equation

$$\sum_{j=0}^{p} \alpha_j y_{t-j} = \sum_{j=0}^{q} \beta_j x_{t-j}, \quad \text{with} \quad \alpha_0 = 1. \tag{1.1}$$

Consider T realisations of the equation (1.1) indexed by $t =\in \{0, \pm 1, \pm 2, \ldots\}$. By associating each equation with the corresponding power z^t of an indeterminate algebraic symbol z and by adding them together, a z-transform polynomial equation is derived that can be denoted by

$$\alpha(z)y(z) = \beta(z)x(z) \quad \text{or, equivalently,} \quad y(z) = \frac{\beta(z)}{\alpha(z)}x(z). \tag{1.2}$$

Here,

$$\begin{aligned}
y(z) &= y_0 + y_1(z + z^{-1}) + y_2(z^2 + z^{-2}) + \cdots, \\
x(z) &= x_0 + x_1(z + z^{-1}) + x_2(z^2 + z^{-2}) + \cdots, \\
\alpha(z) &= 1 + \alpha_1 z + \cdots + \alpha_p z^p \quad \text{and} \\
\beta(z) &= \beta_0 + \beta_1 z + \cdots + \beta_q z^q
\end{aligned} \tag{1.3}$$

are described as the z-transforms of the corresponding sequences.

Expansions of Rational Function

The rational function $\beta(z)/\alpha(z) = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}$ has a series expansion. The method of finding the coefficients of the series expansion can be illustrated by the second-order case:

$$\frac{\beta_0 + \beta_1 z}{1 - \phi_1 z - \phi_2 z^2} = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}. \tag{1.4}$$

We rewrite this equation as

$$\beta_0 + \beta_1 z = \{1 - \phi_1 z - \phi_2 z^2\}\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}. \tag{1.5}$$

Then, by performing the multiplication on the RHS, and by equating the coefficients of the same powers of z on the two sides of the equation, we find that

$$\begin{aligned}
\beta_0 &= \omega_0, & \omega_0 &= \beta_0, \\
\beta_1 &= \omega_1 - \phi_1\omega_0, & \omega_1 &= \beta_1 + \phi_1\omega_0, \\
0 &= \omega_2 - \phi_1\omega_1 - \phi_2\omega_0, & \omega_2 &= \phi_1\omega_1 + \phi_2\omega_0, \\
&\vdots & &\vdots \\
0 &= \omega_n - \phi_1\omega_{n-1} - \phi_2\omega_{n-2}, & \omega_n &= \phi_1\omega_{n-1} + \phi_2\omega_{n-2}.
\end{aligned} \tag{1.6}$$

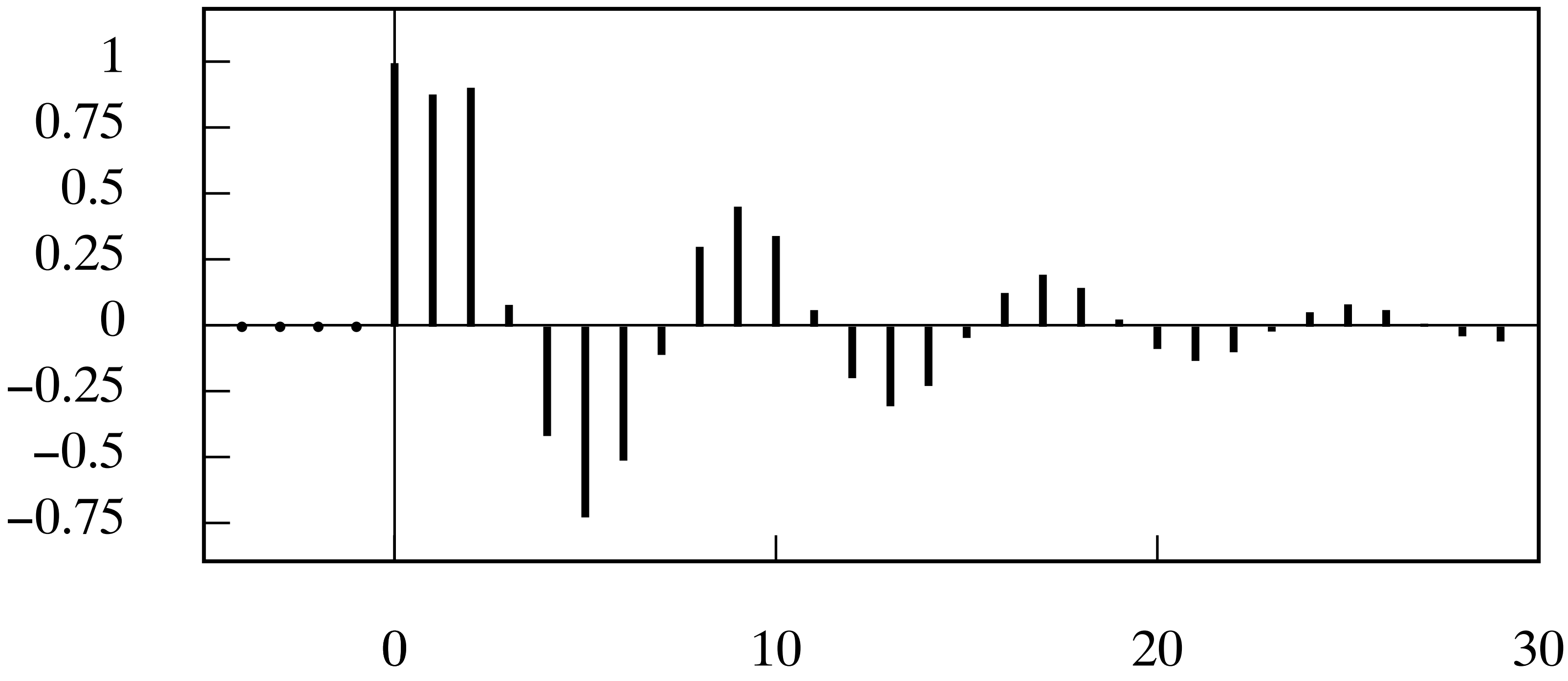

Figure 1. The impulse response of the transfer function $b(z)/a(z)$ with $a(z) = 1.0 - 0.673z + 0.463z^2 + 0.486z^3$ and $b(z) = 1.0 + 0.208z + 0.360z^2$.

Representations via Toeplitz Matrices

The set of T equations can be arrayed in a matrix format as follows:

$$\begin{bmatrix} y_0 & y_{-1} & \dots & y_{-p} \\ y_1 & y_0 & \dots & y_{1-p} \\ \vdots & \vdots & \ddots & \vdots \\ y_p & y_{p-1} & \dots & y_0 \\ \vdots & \vdots & & \vdots \\ y_{T-1} & y_{T-2} & \dots & y_{T-p-1} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_p \end{bmatrix} = \begin{bmatrix} x_0 & x_{-1} & \dots & x_{-q} \\ x_1 & x_0 & \dots & x_{1-q} \\ \vdots & \vdots & \ddots & \vdots \\ x_q & x_{q-1} & \dots & x_0 \\ \vdots & \vdots & & \vdots \\ x_{T-1} & x_{T-2} & \dots & x_{T-q-1} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_q \end{bmatrix}. \tag{1.7}$$

Setting $y_{-1} = \cdots = y_{-p} = 0$ and $x_{-1} = \cdots = x_{-k} = 0$ and extending the matrices on the right creates banded lower-triangular (LT) Toeplitz matrices of order T.

Let $I_T = [e_0, e_1, \ldots, e_{T-1}]$ and define $L_T = [0, e_0, \ldots, e_{T-2}]$, which is a matrix lag operator. Setting $z = L_T$ within $\alpha(z)$ and $\beta(z)$ produces LT Toeplitz matrices $A = \alpha(L_T)$ and $B = \beta(L_T)$. Likewise, $y(z) = y_0 + y_1 z + \cdots + y_{T-1}z^{T-1}$ and $x(z) = x_0 + x_1 z + \cdots + x_{T-1}z^{T-1}$ give $Y = y(L_T)$ and $X = x(L_T)$. Then,

$$YAe_0 = Y\alpha = AYe_0 = Ay \qquad \text{and, likewise,} \qquad XBe_0 = X\beta = BXe_0 = Bx, \tag{1.8}$$

where $\alpha = Ae_0$, $\beta = Be_0$, $y = Ye_0$ and $x = Xe_0$ are the leading columns of the matrices.

Representations via Circulant Matrices

Define $K_T = [e_1, e_2, \ldots, e_{T-1}, e_0]$. This is an orthonormal circulant matrix such that $K_T' K_T = K_T K_T' = I_T$. Its powers form a T-periodic sequence: $K^{T+q} = K^q$ for all q.

The powers $K_T^0 = I_T, K_T, \ldots K_T^{T-1}$ form a basis for the circulant matrices of order T. If $\alpha(z)$ is a polynomial of degree less than T, then there is a corresponding circulant matrix:

$$A = \alpha(K_T) = \alpha_0 I_T + \alpha_1 K_T + \cdots + \alpha_{T-1} K_T^{T-1}. \tag{1.9}$$

An absolutely summable sequence $\{\gamma_i\}$ can also be mapped into a circulant matrix by a process of circular wrapping. Thus, if $\sum |\gamma_i| < \infty$, and if $\gamma(z) = \sum \gamma_j z^j$, then setting $z = K_T$ gives

$$\begin{aligned}\Gamma = \gamma(K_T) &= \left\{\sum_{j=0}^{\infty} \gamma_{jT}\right\} I_T + \left\{\sum_{j=0}^{\infty} \gamma_{(jT+1)}\right\} K_T + \cdots + \left\{\sum_{j=0}^{\infty} \gamma_{(jT+T-1)}\right\} K^{T-1} \\ &= \varphi_0 I_T + \varphi_1 K_T + \cdots + \varphi_{T-1} K_T^{T-1}.\end{aligned} \tag{1.10}$$

Given that $\{\gamma_i\}$ is a convergent sequence, it follows that the sequence of the matrix coefficients $\{\varphi_0, \varphi_1, \ldots, \varphi_{T-1}\}$ converges to $\{\gamma_0, \gamma_1, \ldots, \gamma_{T-1}\}$ as T increases.

If $X = x(K_T)$ and $Y = y(K_T)$ are circulant matrices, then $XY = YX$ is also a circulant matrix. (1.11)

The Spectral Factorisation of a Circulant Matrix

The spectral factorisation of K_T entails the discrete Fourier transform. Define the Fourier matrix

$$U_T = T^{-1/2}[W_T^{jt}; t, j = 0, \ldots, T-1], \tag{1.12}$$

of which the generic element in the jth row and tth column is

$$W_T^{jt} = \exp(-\mathrm{i}2\pi jt/T) = \cos(\omega_j t) - \mathrm{i}\sin(\omega_j t), \qquad \text{where} \quad \omega_j = 2\pi j/T. \tag{1.13}$$

The matrix U_T is a unitary, and it fulfils the condition

$$\bar{U}_T U_T = U_T \bar{U}_T = I_T, \tag{1.14}$$

where $\bar{U}_T = T^{-1/2}[W_T^{-jt}; t, j = 0, \ldots, T-1]$ denotes the conjugate matrix.

The operator K_T can be factorised as

$$K_T = \bar{U}_T D_T U_T = U_T \bar{D}_T \bar{U}_T, \tag{1.15}$$

where

$$D_T = \mathrm{diag}\{1, W, W^2, \ldots, W^{T-1}\} \tag{1.16}$$

is a diagonal matrix whose elements are the T roots of unity, which are on the circumference of the unit circle in the complex plane. Observe that D_T is T-periodic, such that $D_T^{q+T} = D_T^q$, and that $K_T^q = \bar{U}_T D_T^q U_T = U_T \bar{D}_T^q \bar{U}_T$ for any integer q.

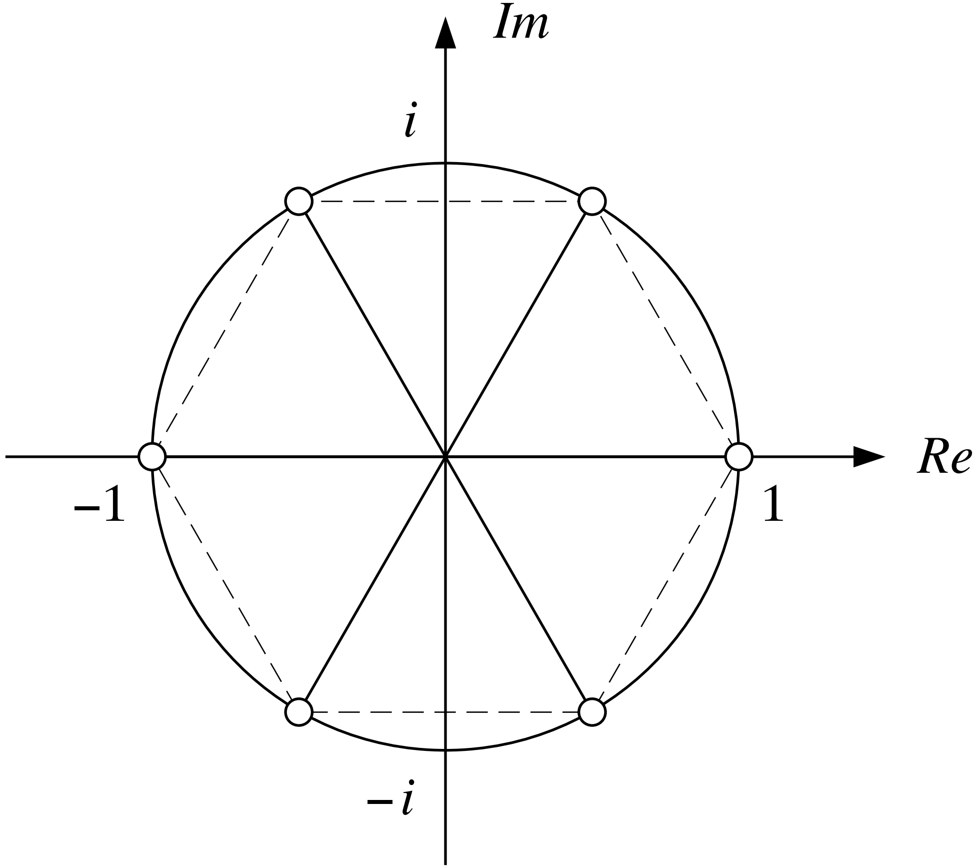

Figure. The 6th roots of unity inscribed in the unit circle.

Linear and Circular Convolutions

If $\psi(j) = \{\psi_j; j = 0 \pm 1, \pm 2, \ldots\}$ are filter coefficients and if $y(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ are data, then the convolution $\psi(j) * y(t) = x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$, which is the filtered sequence, has

$$x_t = \sum_j \psi_j y_{t-j} = \sum_j \psi_{t-j} y_j. \tag{1.17}$$

Convolution is also entailed in generating the coefficients of the product of two polynomials or power series. Define the z-transforms $\psi(z) = \sum_j \psi_j z^j$, $y(z) = \sum_t y_t z^t$ and $x(z) = \sum_t x_t z^t$. Then, in place of the convolution, there is a multiplication or a "modulation":

$$x(z) = \psi(z) y(z). \tag{1.18}$$

The product $X = \Psi Y$ of the LT Toeplitz matrices $\Psi = \psi(L_T)$ and $Y = y(z)$ entails a convolution.

There is also a process of circular convolution, which is applicable to finite sequences. If these are $\{\psi_0, \psi_1, \ldots, \psi_n\}$ and $\{y_0, y_1, \ldots, y_n\}$, then the generic element of their circular convolution is

$$x_t^\circ = \sum_j \psi_j^\circ y_{t-j}^\circ = \sum_t \psi_{j-t}^\circ y_t^\circ, \tag{1.19}$$

wherein $\psi_j^\circ = \psi_{j \bmod n}$ and $y_t^\circ = y_{t \bmod n}$. The product $X^\circ = \Psi^\circ X^\circ = \{\bar{U}\gamma_\psi(D)U\}\{\bar{U}\gamma_x(D)U\} = \bar{U}\{\gamma_\psi(D)\gamma(D)_x\}U$ of the circulant matrices $\Psi^\circ = \psi(K_T)$ and $Y^\circ = y(K_T)$ entails a circular convolution. Also, it entails the frequency-domain modulation $\gamma(D) = \gamma_\psi(D)\gamma(D)_x$.

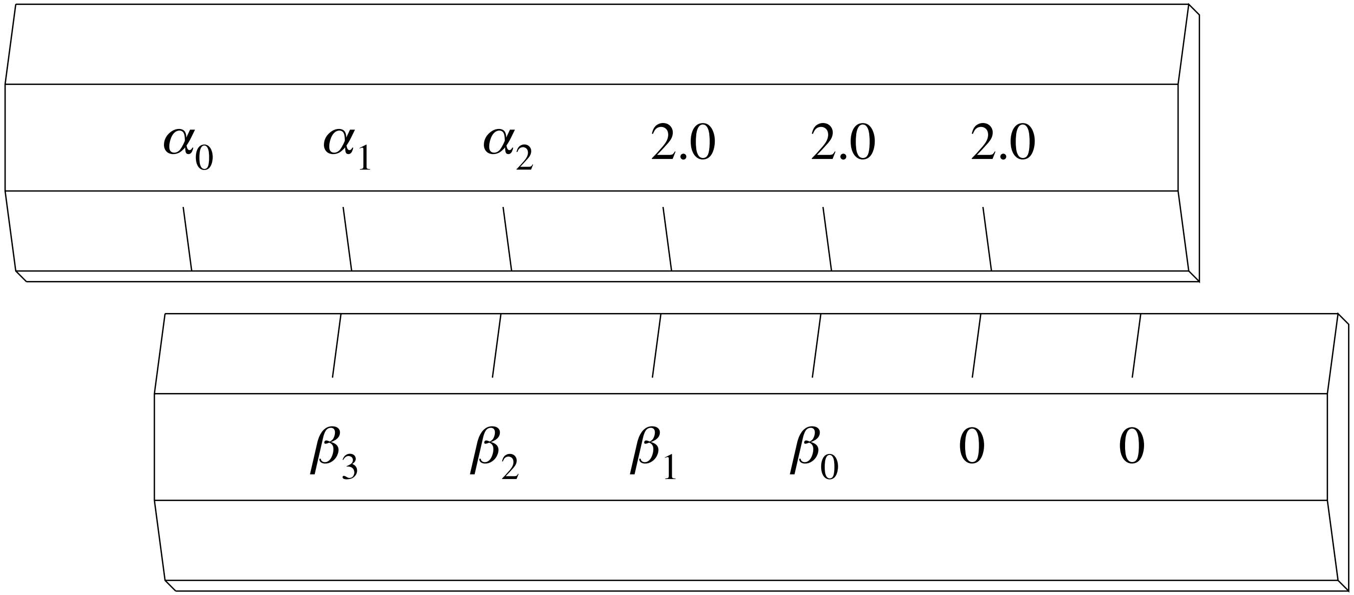

Figure. A method for finding the linear convolution of two sequences. The element $\gamma_4 = \alpha_1\beta_3 + \alpha_2\beta_2$ of the convolution may be formed by multiplying the adjacent elements on the two rulers and by summing their products.

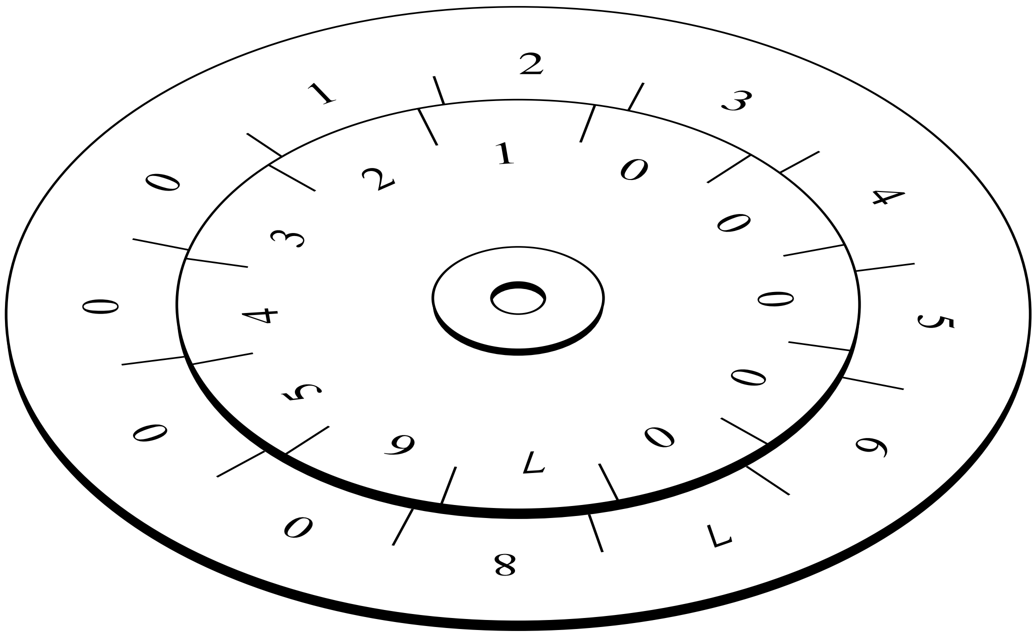

Figure. A device for finding the circular convolution of two sequences. The upper disc is rotated clockwise through successive angles of 30 degrees. Adjacent numbers on the two discs are multiplied and the products are summed to obtain the coefficients of the convolution.