CONCRETE.PAS: A PROGRAM INVESTIGATING DISCRETE AND CONTINUOUS LINEAR DYNAMIC MODELS

1. Introduction

This program investigates the relationship between dynamic linear stochastic models in discrete and continuous time. The discrete-time ARMA model is driven by a white-noise process that is limited in frequency by the Nyquist value of π radians per sample interval. Its direct continuous-time counterpart is a CARMA model, driven by a continuous frequency-limited white-noise process.

The ARMA model may also represent the exact discrete counterpart of a linear stochastic differential equation (LSDE) driven by a continuous-time white-noise process of unlimited frequency comprising the infinitessimal increments of a Wiener process. Such an ARMA model is described as an exact discrete linear model (EDLM).

The program asks the user to specify either a discrete-time ARMA model or a continuous-time LSDE. In the case of an ARMA specification, the program will derive both the corresponding CARMA model and the LSDE model together with their autocovariance functions and their spectral density functions. In the case of an LSDE specification, the program will derive its exact discrete counterpart, which is the EDLM.

2. The Discrete and Continuous Models and their Acronyms

The Discrete-Time Autoregressive Moving-Average Model:

$$(\alpha_0 + \alpha_1 L + \cdots + \alpha_p L^p)y(t) = (\beta_0 + \beta_1 L + \cdots + \beta_q L^q)\varepsilon(t).$$

The Linear Stochastic Differential Equation:

$$(\phi_0 D^p + \phi_1 D^{p-1} + \cdots + \phi_p)y(t) = (\theta_0 D^q + \theta_1 D^{q-1} + \cdots + \theta_q)\zeta(t).$$

Acronyms

ARMA: Autoregressive Moving Average,

LSDE: Linear Stochastic Differential Equation,

CARMA: Continuous-Time Autoregressive Moving Average,

EDLM: Exact Discrete Linear Model.

N.B.

The CARMA models are frequency-limited linear stochastic differential equations corresponding to the ARMA models.

The EDLM models are the discrete-time models that are the exact counterparts of the LSDE models.

3. The Menus of the Program

At the outset, the program displays the following principal menu:

DISCRETE AND CONTINUOUS ARMA MODELS

1. *Get Page Parameters*
2. *Get Model Parameters*
3. *Find the Equivalent Discrete and Continuous Models*
4. *Plot the Autocovariances*
5. *Plot the Spectral Density Function*
6. *Assign Primary Status to a Derived Model*
7. *SAVE and EXIT*

The items of this menu may be accessed successively in the order in which they are listed. The procedures gathered under each menu generate items that are the prerequisites of the procedures of the succeeding menus. However, the menus can be accessed in any order, and requests will be made automatically to supply any prerequisite items that are missing.

For example, if the menu *6. Assign Primary Status to a Derived Model* is accessed immediately on entering the program, then the message *You have not yet specified any model!* will be posted, whereafter the program will return to the main menu.

We proceed to examine each of the items of the main menu in their listed order.

1. Get Page Parameters

The program plots a variety of diagrams on the computer screen and it also generates the corresponding PostScript code, which may be viewed and printed via the GhostScript program or incorporated into a TEX or a LaTEX document.

First, the user is asked whether they require PostScript graphics. If so, then the choice is between the *Textures format* and the *Encapsulated PostScript (EPS) format,* which is the default option.

Next, the user is asked to specify the dimensions of the frame surrounding the graphs. This frame is the bounding box of the graphic. If the only purpose is to view the graphics on the screen, then the maximum dimensions should be chosen. This can be achieved by typing large values in excess of the maximum sizes. Thus, an adequate response to the requests to *Specify the width* and *Specify the height* would be to type 99 in both cases.

2. Get Model Parameters

The program requires the specification of a discrete-time ARMA model or a continuous-time LSDE model.

The $ARMA(p,q)$ model is defined by a rational transfer function $\beta(z)/\alpha(z)$, where the denominator $\alpha(z)$ is an autoregressive polynomial of degree p and $\beta(z)$ is a moving-average polynomial of degree $q < p$.

The LSDE model is defined similarly by a rational transfer function $\theta(s)/\phi(s)$, where $\phi(s)$ is of degree q and $\theta(s)$ is of degree q. It is also required that $q < p$.

There are three ways of specifying the polynomials:

- *Specify the coefficients of the polynomial*

- *Specify the roots of the polynomials in Cartesian form*
- *Specify the roots of the polynomial in polar exponential form*

In each case, the specification is subject to restrictions that ensure that the resulting model fulfils the condition of stability and the condition of invertibility—also described as the miniphase condition.

In the case of the ARMA models, the conditions require that the roots of $\alpha(z^{-1}) = 0$ and $\beta(z^{-1}) = 0$ should fall inside the unit circle. In the case of the LSDE models, the requirements are that the roots of $\phi(s) = 0$ and $\theta(s) = 0$ should lie in the left half-plane.

Unless the coefficients of a stable and invertible model have been generated elsewhere, it is best to specify the models via the roots. In the case of an ARMA model when complex roots are involved, it is easiest to specify the moduli and the arguments of the polar exponential form $\mu = \rho \exp\{\mathrm{i}\omega\}$ via the moduus $\rho < 1$ and the argument $\omega \in [0, 2\pi)$.

In the case of an LSDE model, a Cartesian specification $\kappa = \delta + \mathrm{i}\omega$, in which the real component δ must be negative, may be more appropriate. Nevertheless, the program allows the roots of the LSDE to be specified also in polar form via ρ and ω, which gives rise to the root $\kappa = \ln(\rho) + \mathrm{i}\omega$.

3. Find the Equivalent Discrete and Continuous Models

Given a specified ARMA model, the procedures of this menu will find both the corresponding CARMA model and an LSDE model. The first of the options is

- *Display the coefficients of the equivalent frequency-limited continuous-time CARMA model*

A frequency-limited CARMA model is derived from an ARMA model by a simple translations that is indicated by the principle of impulse invariance; and there is a one-to-one correspondences between the discrete-time ARMA models and their continuous-time CARMA counterparts.

The second option, in the case of a specified ARMA model, is

- *Calculate and display the coefficients of the corresponding LSDE model driven by a Wiener forcing function*

An iterative procedure employing the Nelder-Mead optimisation algorithm is used in finding the LSDE. If, on convergence, the value of criterion function is zero, then the autocovariance functions of the ARMA model and of the LSDE will have identical values at the integer points. In that case, the ARMA model can be described as the exact linear discrete model or EDLM corresponding to the LSDE.

Given a specified LSDE, the program will find the corresponding ARMA model, also describe as the EDLM:

- *Calculate and display the coefficients of the corresponding exact discrete EDLM model*

The autoregressive parameters of the EDLM are inferred directly from those of the LSDE. The moving-average parameters of the EDLM are found via Cramér–Wold decomposition of a discrete autocovariance function sampled from the continuous aurocovariance function of the LSDE at the integer points. (Given that the autoregressive parameters are known, only the moving-average parameters need to be determined via the decomposition.)

4. Plot the Autocovariances

The autocovariance functions of the available discrete and continuous models may be plotted, for a chosen number of integral time lags. The models in question may be the discrete or continuous models that have been specified by the user—described as the primary models—or they may be the derived models, obtained via a discrete–continuous or a continuous–discrete translation.

In the case of an ARMA(p, q) model with $p > q$, the discrete autocovariance sequence is plotted together with a continuous envelope representing the autocovariance function of the corresponding continuous-time CARMA model. Otherwise, if $p \leq q$, only a discrete sequence of autocovariances is plotted.

In case an LSDE model has been derived from an ARMA model, the ARMA autocovariance function and the LSDE autocovariance function will be plotted in succession.

5. Plot the Spectral Density Function

The spectral density functions or 'spectra' of the available discrete and continuous models may be plotted.

The spectrum of an ARMA model and of the associated CARMA model are plotted over the frequency interval $[0, \pi]$. The spectrum of an LSDE model is unbounded in frequency. Therefore, the program affords the possibility of plotting this spectrum on any interval running from 0 up to a maximum of 10π.

In case an LSDE model has been derived from an ARMA model, the ARMA spectrum and the LSDE spectrum will be plotted in succession. The option is available for superimposing the ARMA spectrum upon the LSDE spectrum. In that case, it may be appropriate to plot the LSDE spectrum on the interval $[0, 2\pi]$. The difference between the two spectra will indicate the extent of the aliasing affecting the ARMA model.

6. Assign Primary Status to a Derived Model

The primary model is the parent of the derived models. Primary status can be re-assigned to any available derived model. In the process, the model to which the primary status has been reassigned will become the sole available model.

In the case that both an LSDE and a CARMA model have been derived from an ARMA model, the program will present the following options:

- *Make the LSDE model the primary model*
- *Make the CARMA model the primary model*

When an EDLM has been derived from an LSDE, the only option is

- *Make the EDLM model the primary model*

7. SAVE and EXIT

The program offers the options to *SAVE the parameters* of the available models, to *EXIT the program* or to *RETURN to the main menu.* On saving the parameters, the only option is to return to the main menu, whereafter one might either execute additional functions or else *EXIT the program*

5. The Program in Use

The following listing demonstrates the use of the program in translating an ARMA(2, 1) model to an LSDE(2, 1) model. The process is accompanied by the production of graphs of the associated autocovariance functions and spectral density functions. At the conclusion of the process, the original ARMA model is recovered from the derived LSDE.

The inputs required of the user are indicated by the longrightarrows: $\longrightarrow$.

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

1. Get Page Parameters
$\longrightarrow$ 1

Do you want to output in PostScript?
$\longrightarrow$ N

Specify the width
$\longrightarrow$ 99 {*for maximum width*}

Specify the height
$\longrightarrow$ 99 {*for maximum height*}

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

2. Specify the Parameters of the ARMA Model
$\longrightarrow$ 2

SPECIFY THE MODEL PARAMETERS

Do you wish to specify a Discrete Model or a Continuous Model, D/C?
$\longrightarrow$ D

SPECIFY A DISCRETE-TIME AUTOREGRESSIVE MOVING-AVERAGE MODEL

Specify the Autoregressive Parameters

Do you wish to specify the auxiliary polynomial in polar form, Y/N?
$\longrightarrow$ Y

How many pairs of complex conjugates?
$\longrightarrow$1

Complex conjugate roots

Specify the argument omega[1] in degrees
$\longrightarrow$45

Specify the modulus rho[1]
$\longrightarrow$0.5

Specify the Moving-Averge Parameters

Specify coeff[0] $\longrightarrow$1.0

Specify coeff[1] $\longrightarrow$ -0.5

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

3. Find the Equivalent Discrete and Continuous Models
$\longrightarrow$3

DISCRETE AND CONTINUOUS MODELS

1. Display the Coefficients of the ARMA model
$\longrightarrow$1

The Autoregressive Parameters

$\alpha_0 = 1.0$

$\alpha_1 = -0.7071$

$\alpha_2 = 0.2500$

The Moving-Average Parameters

$\beta_0 = 1.0000$

$\beta_1 = -0.5000$

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

3. Find the Equivalent Discrete and Continuous Models
$\longrightarrow$3

DISCRETE AND CONTINUOUS MODELS

3. Calculate and Display the Coefficient of the Corresponding LSDE Model
$\longrightarrow$3

STARTING VALUES FOR THE MINIMISATION

1. Use the Default Starting Values
$\longrightarrow$1

THE COEFFICIENTS OF THE DERIVED LSDE MODEL

The Autoregressive Parameters

$\phi_0 = 1.0$

$\phi_1 = 1.3863$

$\phi_2 = 1.0973$

The Moving-Average Parameters

$\theta_0 = 1.5012$

$\theta_1 = -0.8905$

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

4. Plot The Aurocovariances
$\longrightarrow$4

THE AUTOCOVARIANCES OF A DISCRETE-TIME ARMA PROCESS

How Many Lagged Autocovariances?
⟶15

The program plots the theoretical autocorrelation function of an ARMA(2, 1) process. To continue, close the graph window and press ⟨RETURN⟩. Then the program will plot the theoretical autocorrelation function of the derived LSDE(2, 1) process.

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

5. Plot The Spectral Density Function
⟶5

The program plots the spectrum of an ARMA(2, 1) process. To continue, close the graph window and press ⟨RETURN⟩.

THE SPECTRUM OF THE DERIVED LSDE MODEL

Do you wish to plot the spectrum of the LSDE, Y/N?
⟶Y

Specify the number of radians in the frequency interval as an
integer multiple of pi radians, not exceeding 10
⟶2

Do you wish to superimpose the ARMA spectrum, Y/N?
⟶Y

The program plots the spectrum of an ARMA process superimposed on the spectrum of an LSDE(2, 1) process. To continue, close the graph window and press ⟨RETURN⟩.

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

6. Assign Primary Status to a Derived Model
⟶6

ASSIGN PRIMARY STATUS TO A DERIVED MODEL

The status is DISCRETE, the primary model is an ARMA model

(L) To make the LSDE the primary model, type L
⟶L

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

3. Find the Equivalent Discrete and Continuous Models
⟶3

DISCRETE AND CONTINUOUS MODELS

2. Calculate and display the coefficients of the corresponding EDLM model
⟶2

THE COEFFICIENTS OF THE DERIVED EDLM MODEL

The Autoregressive Parameters

$\alpha_0 = 1.0$

$\alpha_1 = -0.7071$

$\alpha_2 = 0.2500$

The Moving-Average Parameters

$\beta_0 = 1.0000$

$\beta_1 = -0.5000$

CONCRETE.PAS: Discrete and Continuous Linear Dynamic Models

7. SAVE and EXIT
$\longrightarrow$7

SAVE AND EXIT

3. EXIT the Program
$\longrightarrow$3

Compiling the Program

The Program has been compiled with the *Free Pascal* compiler on a *Windows* platform. *Free Pascal* compilers exist for a wide variety of platforms. The instruction *FPC MAIN* is sufficient for compiling the code that has been provided in the current folder. The code of the program is contained in 15 units, which are compiled in the following order:

1. GLOBALS
2. MATHS
3. UTILS
4. LEGENDS
5. SCREEN
6. POST
7. SPECTRUM
8. CACOVARS
9. DACOVARS
10. ROOTS
11. OPTIMUM
12. CONVERTS
13. ACQUIRE
14. ORGANISE
15. MAIN

The resulted program will be named *MAIN.EXE*.