PROBABILITY THEORY

1. Prove that, if A and B are two events, then the probability that at least one of them will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

China plates that have been fired in a kiln have a probability $P(C) = 1/10$ of being cracked, a probability $P(G) = 1/10$ of being imperfectly glazed and a probability $P(C \cap G) = 1/50$ or being both both cracked and imperfectly glazed. Find the probability $P(C \cup G)$ that a plate will be spoiled.

An inspection passes 90% of all plates. Assuming that a plate will certainly pass the inspection if it is unspoled find the proportion of the spoiled plates which pass the inspection?

Answer: We find that the proportion of spoiled plates is

$$\begin{aligned} P(C \cup G) &= P(C) + P(G) - P(C \cap G) \\ &= \frac{1}{10} + \frac{1}{10} - \frac{1}{50} = \frac{9}{50}. \end{aligned}$$

The regular inspection passses 9/10 of all plates. Let I be the event of a plate passing its inspection. The assumption that a plate will certainly pass the inspection if it is unspoiled is the condition that $P(I|G^c \cap C^c) = 1$. We are told that $P(I) = 9/10$ and we also know that $P(C^c \cap G^c) = 1 - P(C \cup G) = 41/50$. From the equation

$$P(I) = P(I|G \cup C)P(C \cup G) + P(I|G^c \cap C^c)P(G^c \cap C^c),$$

we find that

$$\begin{aligned} P(I|G \cup C) &= \frac{P(I) - P(I|G^c \cap C^c)P(G^c \cap C^c)}{P(G \cup C)} \\ &= \frac{9/10 - 1 \times 41/50}{9/50} = \frac{4}{9}. \end{aligned}$$

2. Show that the probability that an event A or an event B or both will occur is given by the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and specialise this formula for the case (a) when A, B are mutually exclusive events and for the case (b) where A, B are statistically independent events.

Find the probabilities $P(A), P(B)$ when A, B are statistically independent events such that $P(B) = 2P(A)$ and $P(A \cup B) = 5/8$.

Answer: Given that A, B are independent, we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A)P(B) \\ &= 3P(A) + 2\{P(A)\}^2. \end{aligned}$$

Putting x in place of $P(A)$ and taking $P(A \cup B) = 5/8$ gives

$$\frac{5}{8} = 3x - 2x^2 \quad \text{or} \quad 16x^2 - 24x + 5 = 0.$$

The factorisation is $(4x-5)(4x-1) = 0$ which implies that $x = 5/4$ or $x = 1/4$. Discarding the first solution, we find that

$$P(A) = \frac{1}{4} \quad \text{and} \quad P(B) = \frac{1}{2}.$$

3. Establish what is meant by (a) a pair of mutually exclusive events and (b) a pair of statistically independent events. Prove that, if A and B are two events, then the probability of the occurrence of either or both of them is given by

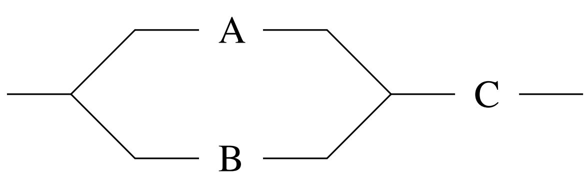
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

A population of fruit flies contains four phenotypes AB, Ab, aB, ab characterised by wing shape (A or a) and eye colour (B or b) which occur in the ratio $9 : 3 : 3 : 1$. If two members of the population are selected at random, what is the probability that they will have the same wing shape (i) given that that they have the same eye colour and (ii) given that they have different eyes colours? Are the two characteristics, wing shape and eye colour, independently assorted?

4. Prove that, if A and B are two events, then the probability that at least one of them will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

The following diagram represents three components in an electrical circuit which are protected by fuses:



For the current to flow, the fuse C must remain intact—the event C—and fuse A or fuse B or both must remain intact—the event $A \cup B$. The probabilities that the fuses will blow are independent and are given by $P(A^c) = 2/3$, $P(B^c) = 1/3$, $P(C^c) = 1/4$. Calculate the probability the the current will flow.

5. Establish what is meant by (a) a pair of mutually exclusive events and (b) a pair of statistically independent events.

Prove, with reference to the axioms of probability and the rules of boolean algebra, that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Use the result above in finding an expression for the probability of the event $A \cup B \cup C$.

On my way to work, I pass three sets of traffic lights which appear to operate independently of each other. They have probabilities of 2/3, 1/3 and 1/2 of showing green. What is the probability that I shall be brought to a halt (a) at all of the lights, (b) at none of the lights and (c) at least once?

6. Establish what is meant by (a) a pair of mutually exclusive events and (b) a pair of statistically independent events.

Prove, with reference to the axioms of probability and the rules of boolean algebra, that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

The probability that a smoker who consumes over 40 cigarettes a day will suffer from chronic respiratory illness is 0.4. The probability that he will suffer from heart disease is 0.5. The probability that he will suffer from at least one of these ailments is 0.6

(i) Find the probability that the smoker will suffer from both ailments,

(ii) Find the probability that he will suffer from heart disease given that he suffers from chronic respiratory illness.

Answer. Let R denote respiratory illness and let H denote heart disease. We have

$$P(R) = \frac{4}{10}, \qquad P(H) = \frac{5}{10}, \qquad P(R \cup H) = \frac{6}{10}.$$

(i) The probability that the smoker will suffer from both ailments is

$$\begin{aligned} P(R \cap H) &= P(R) + P(H) - P(R \cup H) \\ &= \frac{4}{10} + \frac{5}{10} - \frac{6}{10} = \frac{3}{10}. \end{aligned}$$

(ii) The probability that he will suffer from heart disease given that he suffers from chronic respiratory illness is

$$P(H|R) = \frac{P(R \cap H)}{P(R)} = \frac{3}{10}.\frac{10}{4} = \frac{3}{4}.$$

7. Establish what is meant by (a) a pair of mutually exclusive events and (b) a pair of statistically independent events.

Prove, with reference to the axioms of probability and the rules of boolean algebra, that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Use the result above in finding an expression for the probability of the event $A \cup B \cup C$.

A man forgets his banker's card 10% of the time, he forgets his cheque book 5% of the time and he forgets both 2% of the time.

(a) What is the probability that, on any one day, he will have both his banker's card and his cheque book?

(b) What is the probability that he will have his banker's card given that he has his cheque book?

8. Let A and B be two events within the sample space S, for which $P(S) = 1$. Let A^c and B^c be the complements of A and B respectively. Given that $P(A \cup B^c) = 0.3$ and $P(A \cap B) = 0.1$, find $P(B)$.

Answer. Given that $(B \cap A^c) = (A \cup B^c)^c$, we have $P(B \cap A^c) = 1 - P(A \cup B^c) = 0.7$, since $P(A \cup B^c) = 0.3$. Also, $B = (B \cap A^c) \cup (B \cap A)$ is the union of two disjoint sets, so $P(B) = P(B \cap A^c) + P(B \cap A) = 0.7 + 0.1 = 0.8$

Alternative Answer. Consider $B^c \cup (A \cap B) = A \cup B^c$. This is the union of two disjoint sets. Hence $P(B^c) + P(A \cap B) = P(A \cup B^c)$ and, therefore, $P(B^c) = 1 - P(B) = P(A \cup B^c) - P(A \cap B)$, which gives

$$P(B) = 1 - P(A \cup B^c) + P(A \cap B) = 1 - 0.3 + 0.1 = 0.8$$