MOMENT-GENERATING FUNCTIONS

1. Demonstrate how the moments of a random variable x may be obtained from its moment generating function by showing that the rth derivative of $E(e^{xt})$ with respect to t gives the value of $E(x^r)$ at the point where $t = 0$.

Show that the moment generating function of the Poisson p.d.f. $f(x) = e^{-\mu}\mu^x/x!; x \in \{0, 1, 2, \ldots\}$ is given by $M(x,t) = \exp\{-\mu\}\exp\{\mu e^t\}$, and thence find the mean and the variance.

2. Demonstrate how the moments of a random variable may be obtained from the derivatives in respect of t of the function $M(t) = E\{\exp(xt)\}$.

If $x = 1, 2, 3, \ldots$ has the geometric distribution $f(x) = pq^{x-1}$, where $q = 1 - p$, show that the moment generating function is

$$M(t) = \frac{pe^t}{1 - qe^t}.$$

Find $E(x)$.

Answer: The moment generating function of x is

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{xt} pq^{x-1} = \frac{p}{q}\sum_{x=1}^{\infty}\left(qe^t\right)^x \\ &= pe^t \sum_{x=0}^{\infty}\left(qe^t\right)^x = \frac{pe^t}{1 - qe^t}. \end{aligned}$$

To find $E(x)$, we may use the quotient rule to differentiate the expression $M(t)$ with respect to t. This gives

$$\frac{dM(t)}{dt} = \frac{(1 - qe^t)pe^t - pe^t(-qe^t)}{(1 - qe^t)^2}.$$

Setting $t = 0$ gives $E(x) = 1/p$.

3. Let $x_i; i = 1, \ldots, n$ be a set of independent and identically distributed random variables. If the moment generating function of x_i is $M(x_i, t) = E\{\exp(x_i t)\}$ for all i, find the moment generating function for $y = \sum x_i$.

Find the moment generating function of a random variable $x_i = 0, 1$ whose probability density function if $f(x_i) = (1 - p)^{1 - x_i} p^{x_i}$, and thence find the moment generating function of $y = \sum x_i$. Find $E(y)$ and $V(y)$.

4. Demonstrate how the moments of a random variable x may be obtained from the derivatives in respect of t of the function $M(x,t) = E(\exp\{xt\})$

If $x \in \{1, 2, 3 \ldots\}$ has the geometric distribution $f(x) = pq^{x-1}$ where $q = 1 - p$, show that the moment generating function is

$$M(x,t) = \frac{pe^t}{1 - qe^t},$$

and thence find $E(x)$.

5. Demonstrate how the moments of a random variable x—if they exist—may be obtained from its moment generating function by showing that the rth derivative of $E(e^{xt})$ with respect to t, evaluated at the point $t = 0$, gives the value of $E(x^r)$.

Find the moment generating function of $x \sim f(x) = 1$, where $0 < x < 1$, and thereby confirm that $E(x) = \frac{1}{2}$ and $V(x) = \frac{1}{12}$.

6.. Demonstrate how the moments of a random variable x may be obtained from its moment generating function by showing that the rth derivative of $E(e^{xt})$ with respect to t gives the value of $E(x^r)$ at the point where $t = 0$.

Demonstrate that the moment generating function of a sum of independent variables is the product of their individual moment generating functions.

Find the moment generating function of the point binomial

$$f(x;p) = p^x(1-p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution?

Find the variance of $x_1 + x_2$ when $x_1 \sim f(p_1 = 0.25)$ and $x_2 \sim f(p_2 = 0.75)$ are independent point binomials. (MATHSTATS 97)

Answer. It is straightforward to show that $V(x_1) = V(x_2) = pq$ where $p = p_1$ and $q = p_2$ are used for ease of notation. Since x_1, x_2 are statistically independent, it follows that $V(x_1 + x_2) = V(x_1) + V(x_2) = 2pq$.

Alternatively, we may consider the moment generating functions of the two variables x_1 and x_2 which are respectively

$$M_1 = (p + qe^t) \qquad \text{and} \qquad M_2 = (q + pe^t).$$

The moment generating function of their sum $y = x_1 + x_2$ is

$$M = M_1M_2 = (p + qe^t)(q + pe^t) = pq + q^2e^t + p^2e^t + pqe^{2t}.$$

Its first and second derivatives are

$$\frac{dM}{dt} = q^2e^t + p^2e^t + 2pqe^{2t},$$
$$\frac{d^2M}{dt^2} = q^2e^t + p^2e^t + 4pqe^{2t};$$

and setting $t = 0$ gives the following moments:

$$E(y) = q^2 + p^2 + 2pq = 1,$$
$$E(y^2) = q^2 + p^2 + 4pq = 1 + 2pq.$$

It follows that

$$V(y) = E(y^2) - \{E(y^2)\} = 2pq = 0.375$$

7. Find the moment generating function of $x \sim f(x) = 1$, where $0 < x < 1$, and thereby confirm that $E(x) = \frac{1}{2}$ and $V(x) = \frac{1}{12}$.

Answer: The moment generating function is

$$M(x,t) = E(e^{xt}) = \int_0^1 e^{xt}dx$$
$$= \left[\frac{e^{xt}}{t}\right]_0^1 = \frac{e^t}{t} - \frac{1}{t}.$$

But

$$e^t = \frac{t^0}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

so

$$M(x,t) = \left[\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots\right] - \frac{1}{t}$$
$$= 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{24} + \cdots.$$

By the process of differentiating $M(x,t)$ with respect to t and the setting $t = 0$, we get

$$E(x) = \left.\frac{\partial M(x,t)}{\partial t}\right|_{t=0} = \left[\frac{1}{2} + \frac{2t}{3!} + \frac{3t^2}{4!} + \cdots\right]_{t=0} = \frac{1}{2},$$
$$E(x^2) = \left.\frac{\partial^2 M(x,t)}{\partial t^2}\right|_{t=0} = \left[\frac{2}{3!} + \frac{6t}{4!} + \cdots\right]_{t=0} = \frac{1}{3}.$$

Combining these results gives

$$V(x) = E(x^2) - \{E(x)\}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

8. Find the moment generating function of $x \sim f(x) = ae^{-ax}; x \geq 0$.

Answer: The moment generating function is

$$\begin{aligned} M(x,t) = E(e^{xt}) &= \int_0^\infty e^{xt} a e^{-ax} dx = \int_0^\infty a e^{-x(a-t)} dx \\ &= \left[\frac{-ae^{-x(a-t)}}{a-t} \right]_0^\infty = \left[\frac{a}{a-t} \right] = \frac{1}{1-t/a}. \end{aligned}$$

9. Prove that $x \sim f(x) = xe^{-x}; x \geq 0$ has a moment generating function of $1/(1-t)^2$. Hint: Use the change of variable technique to integrate with respect to $w = x(1-t)$ instead of x.

Answer: The moment generating function is

$$M(x,t) = E(e^{xt}) = \int_0^\infty e^{xt} x e^{-x} dx = \int_0^\infty x e^{-x(1-t)} dx.$$

Define $w = x(1-t)$. Then

$$x = \frac{w}{1-t} \quad \text{and} \quad \frac{dx}{dw} = \frac{1}{1-t}.$$

The change of variable technique indicates that

$$\int g(x) dx = \int g\{x(w)\} \frac{dx}{dw},$$

where $g(x) = xe^{-x(1-t)}$. Thus we find that

$$\begin{aligned} M(x,t) &= \int_0^\infty \frac{w}{1-t} e^{-w} \frac{1}{1-t} dw \\ &= \frac{1}{(1-t)^2} \int_0^\infty w e^{-w} dw = \frac{1}{(1-t)^2}. \end{aligned}$$

Here the value of the final integral is unity, since the expression we^{-w}, which is to be found under the integral sign, has the same form as the p.d.f. of x.

To demonstrate directly that the value is unity, we can use the technique of integrating by parts. The formula is

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dv}dx.$$

Within the expression we^{-w}, we take $w = u$ and $e^{-w} = dv/dw$. Then we get

$$\begin{aligned}\int_0^\infty we^{-w}dw &= \left\{\left[-we^{-w}\right]_0^\infty + \int_0^\infty e^{-w}dw\right\}\\ &= \int_0^\infty e^{-w}dw = \left[-e^{-w}\right]_0^\infty = 1.\end{aligned}$$

10. Using the theorem that the moment generating function of a sum of independent variables is the product of their individual moment generating functions, find the m.g.f. of $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$ when $x_1, x_2 \geq 0$ are independent. Can you identify the p.d.f. of $f(x_1 + x_2)$ from this m.g.f.?

Answer: If x_1 and x_2 are independent, then their joint p.d.f. can be written as $f(x_1, x_2) = f(x_1)f(x_2)$; and it follows that

$$\begin{aligned}M(x_1 + x_2, t) &= \int_{x_2}\int_{x_1} e^{(x_1+x_2)t}f(x_1, x_2)dx_1dx_2)\\ &= \int_{x_1} e^{x_1}f(x_1)dx_1 \int_{x_2} e^{x_2}f(x_2)dx_2 = M(x_1, t)M(x_2, t),\end{aligned}$$

or simply that $M(x_1 + x_2, t) = M(x_1, t)M(x_2, t)$. If x_1, x_2 are independent with $x_1 \sim e^{-x_1}$ and $x_2 \sim e^{-x_2}$, then

$$M(x_1, t) = M(x_2, t) = \frac{1}{1-t} \quad \text{and} \quad M(x_1 + x_2, t) = \frac{1}{(1-t)^2}.$$

But, according to the answer to question (3), this implies that

$$f(x_1 + x_2) = (x_1 + x_2)e^{-(x_1+x_2)}.$$

11. Find the moment generating function of the point binomial

$$f(x) = p^x(1-p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution ?

Answer: If $f(x) = p^x(1-p)^{1-x}$ with $x = 0, 1$, then

$$\begin{aligned} M(x,t) &= \sum_{x=0,1} e^{xt} f(x) = e^0 p^0 (1-p) + e^t p (1-p)^0 \\ &= (1-p) + pe^t = q + pe^t. \end{aligned}$$

But the binomial outcome $z = \sum_{i=1}^{n} x_i$ is the sum of n inependent point-binomial outcomes; so it follows that the binomial m.g.f. is

$$M(z,t) = (q + pe^t)^n.$$