1. Demonstrate how the moments of a random variable \( x \) may be obtained from its moment generating function by showing that the \( r \)th derivative of \( E(e^{xt}) \) with respect to \( t \) gives the value of \( E(x^r) \) at the point where \( t = 0 \).

Show that the moment generating function of the Poisson p.d.f. \( f(x) = e^{-\mu} \mu^x / x!; x \in \{0, 1, 2, \ldots\} \) is given by \( M(x, t) = \exp\{-\mu\exp\{\mu e^t\}\} \), and thence find the mean and the variance.

2. Demonstrate how the moments of a random variable may be obtained from the derivatives in respect of \( t \) of the function \( M(t) = E\{\exp(\exp(\text{xt}))\} \).

If \( x = 1, 2, 3, \ldots \) has the geometric distribution \( f(x) = pq^{x-1} \), where \( q = 1 - p \), show that the moment generating function is

\[
M(t) = \frac{pe^t}{1 - qe^t}.
\]

Find \( E(x) \).

**Answer:** The moment generating function of \( x \) is

\[
M(t) = \sum_{x=1}^{\infty} e^{xt} pq^{x-1} = \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x
\]

\[
= \frac{pe^t}{1 - qe^t}.
\]

To find \( E(x) \), we may use the quotient rule to differentiate the expression \( M(t) \) with respect to \( t \). This gives

\[
\frac{dM(t)}{dt} = \frac{(1 - qe^t)pe^t - pe^t(-qe^t)}{(1 - qe^t)^2}.
\]

Setting \( t = 0 \) gives \( E(x) = 1/p \).

3. Let \( x_i; i = 1, \ldots, n \) be a set of independent and identically distributed random variables. If the moment generating function of \( x_i \) is \( M(x_i, t) = E\{\exp(\text{xt})\} \) for all \( i \), find the moment generating function for \( y = \sum x_i \).

Find the moment generating function of a random variable \( x_i = 0, 1 \) whose probability density function if \( f(x_i) = (1 - p)^{1-x_i}p^{x_i} \), and thence find the moment generating function of \( y = \sum x_i \). Find \( E(y) \) and \( V(y) \).
EXERCISES IN STATISTICS

4. Demonstrate how the moments of a random variable $x$ may be obtained from the derivatives in respect of $t$ of the function $M(x, t) = E(\exp\{xt\})$

If $x \in \{1, 2, 3 \ldots\}$ has the geometric distribution $f(x) = pq^{x-1}$ where $q = 1 - p$, show that the moment generating function is

$$M(x, t) = \frac{pe^t}{1 - qe^t},$$

and hence find $E(x)$.

5. Demonstrate how the moments of a random variable $x$—if they exist—may be obtained from its moment generating function by showing that the $r$th derivative of $E(e^{xt})$ with respect to $t$, evaluated at the point $t = 0$, gives the value of $E(x^r)$.

Find the moment generating function of $x \sim f(x) = 1$, where $0 < x < 1$, and thereby confirm that $E(x) = \frac{1}{2}$ and $V(x) = \frac{1}{12}$.

6. Demonstrate how the moments of a random variable $x$ may be obtained from its moment generating function by showing that the $r$th derivative of $E(e^{xt})$ with respect to $t$ gives the value of $E(x^r)$ at the point where $t = 0$.

Demonstrate that the moment generating function of a sum of independent variables is the product of their individual moment generating functions.

Find the moment generating function of the point binomial

$$f(x; p) = p^x(1 - p)^{1-x}$$

where $x = 0, 1$. What is the relationship between this and the m.g.f. of the binomial distribution?

Find the variance of $x_1 + x_2$ when $x_1 \sim f(p_1 = 0.25)$ and $x_2 \sim f(p_2 = 0.75)$ are independent point binomials. (MATHSTATS 97)

**Answer.** It is straightforward to show that $V(x_1) = V(x_2) = pq$ where $p = p_1$ and $q = p_2$ are used for ease of notation. Since $x_1, x_2$ are statistically independent, it follows that $V(x_1 + x_2) = V(x_1) + V(x_2) = 2pq$.

Alternatively, we may consider the moment generating functions of the two variables $x_1$ and $x_2$ which are respectively

$$M_1 = (p + qe^t) \quad \text{and} \quad M_2 = (q + pe^t).$$

The moment generating function of their sum $y = x_1 + x_2$ is

$$M = M_1 M_2 = (p + qe^t)(q + pe^t) = pq + q^2e^t + p^2e^t + pqe^{2t}.$$
MOMENT-GENERATING FUNCTIONS

Its first and second derivatives are
\[
\frac{dM}{dt} = q^2 e^t + p^2 e^t + 2pq e^{2t},
\]
\[
\frac{d^2 M}{dt^2} = q^2 e^t + p^2 e^t + 4pq e^{2t};
\]
and setting \( t = 0 \) gives the following moments:
\[
E(y) = q^2 + p^2 + 2pq = 1,
\]
\[
E(y^2) = q^2 + p^2 + 4pq = 1 + 2pq.
\]
It follows that
\[
V(y) = E(y^2) - \{E(y^2)\} = 2pq = 0.375.
\]

7. Find the moment generating function of \( x \sim f(x) = 1 \), where \( 0 < x < 1 \), and thereby confirm that \( E(x) = \frac{1}{2} \) and \( V(x) = \frac{1}{12} \).

Answer: The moment generating function is
\[
M(x, t) = E(e^{xt}) = \int_0^1 e^{xt} dx
\]
\[
= \left[ \frac{e^{xt}}{t} \right]_0^1 = \frac{e^t}{t} - \frac{1}{t}.
\]
But
\[
e^t = \frac{t^0}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,
\]
so
\[
M(x, t) = \left[ \frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots \right] - \frac{1}{t}
\]
\[
= 1 + \frac{t}{2} + \frac{t^2}{6} + \frac{t^3}{24} + \cdots.
\]
By the process of differentiating \( M(x, t) \) with respect to \( t \) and the setting \( t = 0 \), we get
\[
E(x) = \frac{\partial M(x, t)}{\partial t} \bigg|_{t=0} = \left[ \frac{1}{2} + \frac{2t}{3!} + \frac{3t^2}{4!} + \cdots \right]_{t=0} = \frac{1}{2},
\]
\[
E(x^2) = \frac{\partial^2 M(x, t)}{\partial t^2} \bigg|_{t=0} = \left[ \frac{2}{3!} + \frac{6t}{4!} + \cdots \right]_{t=0} = \frac{1}{3}.
\]
Combining these results gives

\[ V(x) = E(x^2) - \{ E(x) \}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \]

8. Find the moment generating function of \( x \sim f(x) = ae^{-ax}; x \geq 0. \)

**Answer:** The moment generating function is

\[
M(x, t) = E(e^{xt}) = \int_0^\infty e^{xt}ae^{-ax}dx = \int_0^\infty ae^{-(a-t)x}dx
\]

\[
= \left[-ae^{-(a-t)x}\right]_0^\infty = \left[\frac{a}{a-t}\right] = \frac{1}{1-t/a}.
\]

9. Prove that \( x \sim f(x) = xe^{-x}; x \geq 0 \) has a moment generating function of \( 1/(1-t)^2. \) Hint: Use the change of variable technique to integrate with respect to \( w = x(1-t) \) instead of \( x. \)

**Answer:** The moment generating function is

\[
M(x, t) = E(e^{xt}) = \int_0^\infty e^{xt}xe^{-x}dx = \int_0^\infty xe^{-(1-t)x}dx.
\]

Define \( w = x(1-t). \) Then

\[
x = \frac{w}{1-t} \quad \text{and} \quad \frac{dx}{dw} = \frac{1}{1-t}.
\]

The change of variable technique indicates that

\[
\int g(x)dx = \int g\{x(w)\}\frac{dx}{dw},
\]

where \( g(x) = xe^{-(1-t)x}. \) Thus we find that

\[
M(x, t) = \int_0^\infty \frac{w}{1-t}e^{-w}\frac{1}{1-t}dw
\]

\[
= \frac{1}{(1-t)^2} \int_0^\infty we^{-w}dw = \frac{1}{(1-t)^2}.
\]

Here the value of the final integral is unity, since the expression \( we^{-w} \), which is to be found under the integral sign, has the same form as the p.d.f. of \( x. \)
MOMENT-GENERATING FUNCTIONS

To demonstrate directly that the value is unity, we can use the technique of integrating by parts. The formula is

\[ \int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx. \]

Within the expression \( we^{-w} \), we take \( w = u \) and \( e^{-w} = dv/dw \). Then we get

\[ \int_{0}^{\infty} we^{-w} dw = \left\{ [-we^{-w}]_{0}^{\infty} + \int_{0}^{\infty} e^{-w} dw \right\} \]

\[ = \int_{0}^{\infty} e^{-w} dw = \left[ -e^{-w} \right]_{0}^{\infty} = 1. \]

10. Using the theorem that the moment generating function of a sum of independent variables is the product of their individual moment generating functions, find the m.g.f. of \( x_1 \sim e^{-x_1} \) and \( x_2 \sim e^{-x_2} \) when \( x_1, x_2 \geq 0 \) are independent. Can you identify the p.d.f. of \( f(x_1 + x_2) \) from this m.g.f.?

Answer: If \( x_1 \) and \( x_2 \) are independent, then their joint p.d.f. can be written as \( f(x_1, x_2) = f(x_1)f(x_2) \); and it follows that

\[ M(x_1 + x_2, t) = \int_{x_2} e^{(x_1+x_2)t} f(x_1, x_2) dx_1 dx_2 \]

\[ = \int_{x_1} e^{x_1} f(x_1) dx_1 \int_{x_2} e^{x_2} f(x_2) dx_2 = M(x_1, t)M(x_2, t), \]

or simply that \( M(x_1 + x_2, t) = M(x_1, t)M(x_2, t) \). If \( x_1, x_2 \) are independent with \( x_1 \sim e^{-x_1} \) and \( x_2 \sim e^{-x_2} \), then

\[ M(x_1, t) = M(x_2, t) = \frac{1}{1-t} \quad \text{and} \quad M(x_1 + x_2, t) = \frac{1}{(1-t)^2}. \]

But, according to the answer to question (3), this implies that

\[ f(x_1 + x_2) = (x_1 + x_2)e^{-(x_1+x_2)}. \]

11. Find the moment generating function of the point binomial

\[ f(x) = p^x (1-p)^{1-x} \]

where \( x = 0, 1 \). What is the relationship between this and the m.g.f. of the binomial distribution?
Answer: If \( f(x) = p^x (1-p)^{1-x} \) with \( x = 0, 1 \), then

\[
M(x, t) = \sum_{x=0,1} e^{xt} f(x) = e^0 p^0 (1-p) + e^t p (1-p)^0 = (1-p) + pe^t = q + pe^t.
\]

But the binomial outcome \( z = \sum_{i=1}^n x_i \) is the sum of \( n \) independent point-binomial outcomes; so it follows that the binomial m.g.f. is

\[
M(z, t) = (q + pe^t)^n.
\]