CHI-SQUARE TEST OF GOODNESS OF FIT

1. A gambler plays a game that involves throwing 3 dice in a succession of trials. His winnings are directly proportional to the number of sixes recorded. If the dice are fair, what is the probability distribution that governs the outcome of each throw?

The frequencies of the sixes observed in 100 trials are recorded, together with their expected values, in the following table:

Number of sixes	Expected Count	Observed Count
0	58	47
1	34.5	35
2	7	15
3	0.5	3

You are asked to assess whether it is likely that the dice have been unfairly weighted, using a chi-square test of goodness of fit.

Answer. The number of sixes x in a fair trial has a Binomial distribution:

$$b(n = 3, p = 1/6) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \frac{3!}{(3-x)!x!} \left(\frac{1}{5}\right)^x \left(\frac{5}{6}\right)^{n-x}.$$

We have

$$\begin{aligned}
b(x = 0) &= \left(\frac{5}{6}\right)^3 = \frac{125}{216} = 0.579, \\
b(x = 1) &= 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = \frac{75}{216} = 0.347, \\
b(x = 2) &= 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{5}{215} = 0.069, \\
b(x = 3) &= \left(\frac{1}{6}\right)^3 = \frac{1}{216} = 0.0046.
\end{aligned}$$

The Chi-square statistic takes the form of

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{observed}}.$$

We calculate that

$$\begin{aligned}
\chi^2 &= \frac{(47-58)^2}{58} + \frac{(35-34.5)^2}{58} + \frac{(15-7)^2}{7} + \frac{(3-0.5)^2}{0.5} \\
&= 2.08 + 0.007 + 9.14 + 12.5 = 23.727.
\end{aligned}$$

From the table, the 5 percent critical value of the χ^2 of three degrees of freedom is 7.815. Therefore, there is evidence that the dice are unfairly weighted.

2. In an experiment in breeding mice, a geneticist has obtained 120 brown mice with pink eyes, 48 brown mice with brown eyes, 36 white mice with pink eyes and 13 white mice with brown eyes. Theory predicts that these types of mice should be obtained in the ratios 9 : 3 : 3 : 1. Test the compatibility of the data with theory, using a 5% critical value.

Answer. The following table records the observed frequencies in its first row and the frequencies expected under the null hypothesis H_0 in its second row:

observed	120	48	36	13
expected	122	41	41	14

The Chi-square statistic takes the form of

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{observed}}.$$

We calculate that

$$\begin{aligned}\chi^2 &= \frac{(120-122)^2}{122} + \frac{(48-41)^2}{41} + \frac{(36-41)^2}{41} + \frac{(13-14)^2}{14} \\ &= 0.0328 + 1.1957 + 0.6098 + 0.0714 = 1.9097.\end{aligned}$$

From the table, the 5 percent critical value of the χ^2 of three degrees of freedom is 7.815. Therefore, the hypothesis can be maintained.