CHANGE-OF-VARIABLE TECHNIQUE

1. Let x be a continuous random variable with a probability density function $f(x)$ and let $y = y(x)$ be a monotonic transformation. Describe how the probability density function of y is derived if $f(x)$ is known, taking care to distinguish the case where $y = y(x)$ is a positive transformation from the case where it is a negative transformation.

The levels of personal bank acounts are distributed normally with a mean of £690 and a standard deviation of £30. What is the probability that an individual will have more than £720 if it is known that he has more than £700?

2. Let x be a continuously distributed random variable with a probability density function $f(x)$, and let $y = y(x)$ be a monotonic transformation. Describe how the probability density function of y is derived if (x) is known, taking care to distinguish the case where $y = y(x)$ is a positive monotonic transformation from the case where it is a negative transformation.

Let x have the probability density function $f(x) = 1; 0x1$. Find the probability density fuction of $y = -2 \log x$.

Answer: We have $y = y(x) = -2 \log x$ and $x = x(y) = e^{-y/2}$ Hence

$$\frac{dx}{dy} = -\frac{1}{2}e^{-\frac{1}{2}y} \quad \text{and, therefore,}$$

$$g(y) = f\{x(y)\}\left|\frac{dx}{dy}\right| = \frac{1}{2}e^{-\frac{1}{2}y}$$

is the pdf of y.

3. Let $x \sim f(x)$ be a continuous random variable and let $y = y(x)$ be a monotonic transformation. Describe how the probability function of y is derived if $f(x)$ is known, taking care to distinguish the case where $y = y(x)$ is a positive transformation from the case where it is a negative transformation. Find the probability density functions of (a) $2x + 1$ and (b) $2x^2 + 1$.

4. Let x be a continuously distributed random variable with a probability density function $f(x)$, and let $y = y(x)$ be a monotonic transformation. Describe how the probability density function of y is derived if $f(x)$ is known, taking care to distinguish the case where $y = y(x)$ is a positive transformation from the case where it is a negative transformation.

In a statistical model, the time which elapses between the start of a fire and the beginning of the action which extinguishes it is distributed according the function

$$f(t) = ae^{-at}.$$

The costs of the fire damage are deemed to be proportional to $\sqrt{t}$. Find a distribution from which one might infer the expected cost of the fire damage. Comment on the adequacy of this model.

5. Let x be a continuous random variable with a probability density function $f(x)$ and let $y = y(x)$ be a monotonic transformation. Describe how the probability density function of y is derived if $f(x)$ is known, taking care to distinguish the case where $y = y(x)$ is a positive transformation from the case where it is a negative transformation.

The levels of personal bank accounts are distributed normally with a mean of £690 and a standard deviation of £90. What is the probability that an individual will have more than £780 if it is known that he has more than £720?

Answer. Bank accounts are distributed as $N(\mu = 690, \sigma = 90)$. We have to evaluate

$$P(x > 780|x > 720) = \frac{P(x > 780)}{P(x > 720)}.$$

We begin by finding equivalent events in term of the standard normal variate $z = (x - \mu)/\sigma = (x - 690)/90$. We have the following equivalences:

$$x > 780 \Longleftrightarrow z > \frac{780 - 690}{90} = 1,$$

$$x > 720 \Longleftrightarrow z > \frac{720 - 690}{90} = \frac{1}{3}.$$

From tables of the standard normal, we find that

$$\begin{aligned} P(z > 1) &= 1 - (0.5 + 0.3413) \\ &= 0.1587. \end{aligned}$$

and that

$$\begin{aligned} P(z > \tfrac{1}{3}) &= 1 - (0.5 + 0.1293) \\ &= 0.3707. \end{aligned}$$

Therefore

$$\begin{aligned} P(x > 780|x > 720) &= \frac{P(x > 780)}{P(x > 720)} \\ &= \frac{P(z > 1)}{P(z > \frac{1}{3})} = \frac{0.1587}{0.4707} \simeq \frac{1}{2}. \end{aligned}$$