BINOMIAL THEOREM

1. Derive the binomial distribution by considering the sum of the outcomes $x_i; i = 1, \ldots, n$ of n independent trials where $P(x_i = 1) = p$ and $P(x_i = 0) = 1 - p$ for all i.

A particle moves between adjacent nodes of a diagonal lattice. The nodes correspond to the integer points (x, y) for which $x + y$ is even. If the particle's position at time t is (x, y), then its position at time $t + 1$ is governed by the following probabilities: $P(x-1, y+1) = P(x+1, y+1) = P(x+1, y-1) = P(x-1, y-1) = 1/4$.

Assuming that the particle is at the origin at time $t = 0$, find the probability that, at time $t = 4$, it will be at one of the points $(-2, 2)$, $(2, 2)$, $(2, -2)$ or $(-2, -2)$, which represent the vertices of a square with sides of 4 units which is centred at the origin. (MATHSTAT 97)

Answer. The probability which must be assessed may be denoted by

$$\begin{aligned} &P\big\{(2,2) \cup (2,-2) \cup (-2,2) \cup (-2,-2) \\ &\quad = P(2,2) + P(2,-2) + P(-2,2) + P(-2,-2), \end{aligned}$$

where the equality follows from the fact that the events in question are mutually exclusive. The probability of the generic event may be written as $P(x, y) = P(x)P(y)$ in consequence of the fact that the horizontal and vertical components of movements between adjacent nodes of the lattice are statistically independent. Next observe that

$$P(x = 2) = P(x = -2) = P(y = 2) = P(y = -2) = 1/4.$$

In particular, the outcome $x = 2$, which results from one movement to the left and three to the right, has a probability which is given by

$$\begin{aligned} b(z = 1; n = 4, p = 1/2) &= b(z = 3; n = 4, p = 1/2) \\ &= 4\,(1/2)^4 = 1/4, \end{aligned}$$

where $b(z; n, p)$ stands for the binomial p.d.f. It follow, for example, that

$$P(x = 2, y = 2) = P(x = 2)P(y = 2) = 1/16.$$

Hence the probability that the particle will be at one of the four vertices is

$$P\big\{(2,2) \cup (2,-2) \cup (-2,2) \cup (-2,-2) = 4 \times (1/16) = 1/4.$$

2. Derive, from first principles, the function expressing the probability of obtaining x successes in n independent trials when the probability of a success in any trial is p.

On the third floor of the Metropolitan Hotel there are six guest rooms but only four bathrooms. On average, two guests in five require a morning bath. Calculate the probability that, on a morning when all the guest rooms are occupied, some of the bathrooms will have to be used more than once.

Answer: The probability that x out of 6 guests will take baths is given by

$$b\Big(x; n = 6, p = \frac{4}{10}\Big) = \frac{6!}{(6-x)!x!}\left(\frac{4}{10}\right)^x\left(\frac{6}{10}\right)^{6-x}.$$

The probability that bathrooms will have to be used more than once is

$$\begin{aligned} b(5) + b(6) &= 6\left(\frac{4}{10}\right)^5\left(\frac{6}{10}\right) + \left(\frac{4}{10}\right)^6 \\ &= \frac{4^5}{10^6}(36 + 4) = 0.040960. \end{aligned}$$

3. Derive the binomial distribution by considering the sum of the outcomes $x_i; i = 1, \ldots n$ of n independent trials where, in each trial, the probability of a success is p and that of a failure is $1 - p$.

Ten jurors are selected from a long list of names of which one third belong to women. The selection is regarded as unsatisfactory if either sex is represented by less than three people. Find the probability that the selection will be unsatisfactory.

Answer: The probability of a juror being a woman is 1/3. The number x of the women on the jury has a binomial distribution:

$$x \sim b\left(x, n = 10, p = \tfrac{1}{3}\right).$$

The selection is unsatisfactory if $x \in \{0, 1, 2, 8, 9, 10\}$, and the probability of this event is

$$\begin{aligned} &b(0) + b(1) + b(2) + b(8) + b(9) + b(10) \\ &= \left(\frac{2}{3}\right)^{10} + 10\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^9 + \frac{10\times 9}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^8 \\ &\quad + \frac{10\times 9}{2}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^2 + 10\left(\frac{1}{3}\right)^9\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^{10} \\ &= \frac{17,856}{59,049} \simeq \frac{3}{10}. \end{aligned}$$

4. Derive, from first principles, the function expressing the probability of obtaining x successes in n independent trials when the probability of a success in any trial is p.

In the forecourt of the Basingstoke office of the Digital Equipment Corporation there are four parking spaces which are reserved for the sales force. The probability that a salesman will return to the office by lunchtime is 2/5. Given that there are six salesmen on the rota, what is the probability that some of the salesmen will have to leave their cars in the main car park at lunchtime? (MATHSTAT 96)

Answer: The probability that x out of 6 salesman will return by lunchtime is given

$$b\Big(x; n=6, p=\frac{4}{10}\Big) = \frac{6!}{(6-x)!x!}\left(\frac{4}{10}\right)^x\left(\frac{6}{10}\right)^{6-x}.$$

The probability that some of the salesmen will not find a parking space is

$$\begin{aligned} b(5)+b(6) &= 6\left(\frac{4}{10}\right)^5\left(\frac{6}{10}\right)+\left(\frac{4}{10}\right)^6 \\ &= \frac{4^5}{10^6}(36+4) = 0.040960. \end{aligned}$$

5. Derive, from first principles, the function expressing the probability of obtaining x successes in n independent trials when the probability of a success in any trial is p.

On the shuttle service from New York to Boston, an air passenger who buys an open ticket, which is the more expensive variety, can reserve a seat on any flight, but he looses nothing if he fails to catch that flight. On average, only 75% of those who make such reservations actually catch the flight. A booking clerk has accepted ten reservations but has only eight vacant seats. What is the probability that persons who have reserved seats will be made to catch a later flight?

Answer. The probability that x out of 10 people will claim their seats is given

$$b\left(x; n=10, p=\frac{3}{4}\right) = \frac{10!}{(10-x)!x!}\left(\frac{3}{4}\right)^x\left(\frac{1}{4}\right)^{10-x}.$$

The probability that some of these will be made to catch a later flight is

$$\begin{aligned} b(9)+b(10) &= 10\left(\frac{3}{4}\right)^9\left(\frac{1}{4}\right)+\left(\frac{3}{4}\right)^{10} \\ &= \frac{3^9}{4^{10}}(10+3) = ?. \end{aligned}$$

6. Derive, from first principles, the function expressing the probability of obtaining x successes in n independent trials when the probability of a success in any trial is p.

On my journey home, I encounter six sets of traffic lights at widely spaced intervals. They are all timed to give 60 seconds of green and 40 seconds in total of red and amber; and, unless they show green, I am obliged to come to a halt. Assuming that my arrival time at the lights is uniformly random over a wide interval, find the probability that I will not be delayed by more than 3 sets of lights. What is the expected value per journey of the overall delay at traffic lights?

Answer. We are given

$$P(\text{STOP}) = \frac{4}{10} = p, \qquad P(\text{GO}) = \frac{6}{10} = 1 - p,$$
$$\text{Number of Lights} = 6 = n,$$
$$\text{Number of Stoppages} = x \sim b(n, p) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}.$$

The Probability of not being delayed at more than three sets of lights is

$$b(0) + b(1) + b(2) + b(3) = 1 - b(4) + b(5) + b(6)$$

where

$$b(4) = \frac{6.5}{2.1} \left(\frac{4}{10}\right)^4 \left(\frac{6}{10}\right)^2$$
$$b(5) = 6 \left(\frac{4}{10}\right)^5 \left(\frac{6}{10}\right)$$
$$b(6) = \left(\frac{4}{10}\right)^6.$$

Hence

$$\begin{aligned} b(4) + b(5) + b(6) &= \left(\frac{4}{10}\right)^4 \left\{\frac{15 \times 36}{100} + \frac{36 \times 4}{100} + \frac{16}{100}\right\} \\ &= \left(\frac{4}{10}\right)^4 \left\{\frac{540 + 144 + 16}{100}\right\} \\ &= \frac{256 \times 700}{10^6} = 0.1792, \end{aligned}$$

and

$$1 - b(4) + b(5) + b(6) = 0.8208.$$

$$\text{Expected number of delays} = np = 6.\frac{4}{10} = 2.4,$$
$$\text{Average delay at the lights} = 20\text{secs},$$
$$\text{Overall expected delay} = 48\text{secs}.$$

7. On average, 40% of those who reserve seats by telephone actually arrive at the theatre. A booking clerk has accepted six telephone reservations but has only four vacant seats. What is the probability that persons who have reserved seats will not be accommodated?

8. A horticulturist considers that a batch of seeds is worth sowing if 50% of the resulting flowers are going to be pure white. To test the worth of a particular batch, he sows eight seeds with the intention of sowing the remainder if at least four of the eight plants have white flowers. Find the probability of his making a wrong decision (a) if 25% of the seeds are of the white variety, and (b) if 75% of the seeds are of the white variety.

Answer Let p be the probability that a seed selected at random will produced a white flower, and let x be the number of white flowers in the test sample of $n = 8$ seeds. We may assume that

$$x \sim b(n,p) = \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}.$$

(a) Let $p = 1/4$. The wrong decision, which is to sow the seeds, is made if $x \in \{4,5,6,7,8\}$. Observe that $P(x \in \{4,5,6,7,8\}) = 1 - P(x \in \{0,1,2,3,\})$. Then

$$P(x \in \{0,1,2,3,\}) = \sum_{x=0}^{4} \frac{n!}{(n-x)!x!}p^x(1-p)^{n-x}$$
$$= \left(\frac{3}{4}\right)^8 + 8\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^7 + \frac{8.7}{2.1}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^6 + \frac{8.7.6}{3.2.1}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^5$$
$$= \left(\frac{1}{4}\right)^8 \left\{3^8 + 8.3^7 + 4.7.3^6 + 8.7.3^5\right\}.$$

We have

$$3^5 = 243, \quad 3^6 = 729, \quad 3^7 = 2187, 3^8 = 6561, \quad \text{and} \quad 4^8 = 65536.$$

Therefore

$$P(x \in \{0,1,2,3,\}) = \frac{1}{65536}\left\{6561 + (8 \times 2187) + (28 \times 729) + (56 \times 243)\right\}$$
$$= 0.8877,$$

and

$$P(x \in \{4,5,6,7,8\}) = 1 - 0.8877 = 0.1153$$