BAYES THEOREM

1. Show how Bayes' theorem is used to obtain the posterior likelihood of an hypothesis from the prior likelihood when an event has occured which throws some light on the hypothesis.

Usually I wash the breakfast dishes before leaving for the office. If I do not do so, then my wife will do so nine times out of ten if she returns from work first; but there is only a one-in-ten chance that my children will do so if they return from school before I get home.

This morning I failed to wash the dishes. Also my wife told me that there is a fifty-fifty chance that she would return later than me. When I returned home, I discovered that the dishes had been washed, and I could hear that my children were at home. What were the chances that my wife was also at home?

2. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where $P(E) = \sum_i P(E|H_i)P(H_i)$.

Until the early sixties, the dawn train used to collect any milk churns that were left on the platform of Worplesham station on weekdays. Churns would be left on three of the five days. Imagine that I have arrived at the station not knowing the exact time and thinking that there is a fifty-fifty chance that I have missed the train. Then I notice that there are no milk churns on the platform. How should I reassess the chances that I have missed the train?

Answer: Let T be the event that the train has already passed and let N be the event of there being no milk churns on the platform when I arrive. We have

$$P(T|N) = \frac{P(N|T)P(T)}{P(N)}$$

with

$$P(N) = P(N|T)P(T) + P(N|T^c)P(T^c).$$

We take

$$P(N|T) = 1, \quad P(T) = P(T^c) = \frac{1}{2} \quad \text{and} \quad P(N|T^c) = \frac{2}{5}.$$

Then

$$P(N) = 1.\frac{1}{2} + \frac{2}{5}.\frac{1}{2} = \frac{7}{10}$$

$$\text{and} \quad P(T|N) = 1.\frac{1}{2}.\frac{10}{7} = \frac{5}{7}.$$

3. Let H denote an hypothesis and let E denote an event. Show that

$$P(H_i|E) = \frac{P(E|H_i)H(H_i)}{P(E)}$$

where $P(E) = \sum_i P(E|H_i)P(H_i)$.

Of the tins in a paint store, 60% contain Brilliant White paint and 40% contain Off-White paint. The tins are unmarked. However, $\frac{2}{3}$ of the Brilliant White paint and $\frac{1}{3}$ of the Off-White paint is delivered in tins without handles.

(a) What is the probability that a tin will contain Brilliant White paint given that it has a handle?

(b) What is the probability that, by picking a tin with a handle and another without a handle, the storekeeper will select one of each colour?

Answer: Let the event of selecting brilliant white be denoted by B and that of selecting off-white by W. Let H denote a tin with and handle and H^c a tin without one. We have

$$P(B) = \frac{6}{10} \qquad P(W) = \frac{4}{10}$$

$$P(H^c|B) = \frac{2}{3} \qquad P(H^c|W) = \frac{1}{3}$$

$$P(H|B) = \frac{1}{3} \qquad P(H|W) = \frac{2}{3}$$

Also we have

$$\begin{aligned} P(H) &= P(H|B)P(B) + P(H|W)P(W) \\ &= \left(\frac{1}{3} \times \frac{6}{10}\right) + \left(\frac{2}{3} \times \frac{4}{10}\right) = \frac{14}{30} \\ P(H^c) &= P(H^c|B)P(B) + P(H^c|W)P(W) \\ &= \left(\frac{2}{3} \times \frac{6}{10}\right) + \left(\frac{1}{3} \times \frac{4}{10}\right) = \frac{16}{30} \end{aligned}$$

Therefore

$$P(B|H) = \frac{P(H|B)P(B)}{P(H)} = \frac{1}{3} \times \frac{6}{10} \times \frac{30}{14} = \frac{3}{7}$$

and

$$P(B|H^c) = \frac{P(H^c|B)P(B)}{P(H^c)} = \frac{2}{3} \times \frac{6}{10} \times \frac{30}{16} = \frac{3}{4}$$

The event of getting a tin of each colour when one with a handle and one without are selected is

$$\{(B \cap H) \cap (W \cap H^c)\} \cup \{(B \cap H^c) \cap (W \cap H)\}$$

and the corresponding probability is

$$P(B|H)P(W|H^c) + P(B|H^c)P(W|H)$$

With

$$P(W|H^c) = 1 - P(B|H^c) = \frac{1}{4} \quad \text{and}$$
$$P(W|H) = 1 - P(B|H) = \frac{4}{7}$$

This gives

$$P(B|H)P(W|H^c) + P(B|H^c)P(W|H) = \left(\frac{3}{7} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{4}{7}\right) = \frac{15}{28}$$

4. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where

$$P(E) = \sum_i P(E|H_i)P(H_i).$$

A sensitive fire alarm has been installed in a petrol refinery. It can always be relied upon to detect fire, but it often gives false alarms. The probability of a fire during a certain period is 2/5 and the probability that the alarm will sound spontaneously is 1/3. If the alarm does sound in that period, what is the probability that there is a genuine fire?

Answer: Let A be the event of the alarm sounding and let F be the event of a fire. We have

$$P(F|A) = \frac{P(A|F)P(F)}{P(A)}$$

with

$$P(A) = P(A|F)P(F) + P(A|F^c)P(F^c).$$

We take

$$P(A|F) = 1, \quad P(A|F^c) = \frac{1}{3}, \quad P(F) = \frac{2}{5} \quad \text{and} \quad P(F^c) = \frac{3}{5}.$$

Then

$$P(A) = 1 \times \frac{2}{5} + \frac{1}{3} \times \frac{3}{5} = \frac{3}{5}$$

$$\text{and} \quad P(F|A) = 1 \times \frac{2}{5} \times \frac{5}{3} = \frac{2}{3}.$$

5. Let H_1 and H_2 be the only possible explanations of an event E. Show that

$$P(H_1|E) = \frac{P(E|H_1)P(H_1)}{P(E)},$$

where

$$P(E) = P(E|H_1)P(H_1) + P(E|H_2)P(H_2).$$

The lights on the Christmas tree have failed; and it seems equally probable that either the fuse in the plug has blown or that one of the lights, which are wired in series, has blown. I have some spare fuses of which I suspect that 20% are dud. In an attempt to diagnose the fault, I fit a new fuse, but still the lights do not work. How should I now assess the chances that a bulb has blown?

Answer. Let F denote the failure of the original fuse, let L denote the blowing of a bulb and let E denote the result of fitting a new fuse which is the continued failure of the lights.

We have

$$P(F) = \frac{1}{2}, \qquad P(L) = \frac{1}{2}, \qquad P(E|L) = 1, \qquad P(E|F) = \frac{2}{5}.$$

Also

$$\begin{aligned} P(E) &= P(E|L)P(L) + (E|F)P(F) \\ &= \left(1 \times \frac{1}{2}\right) + \left(\frac{2}{5} \times \frac{1}{2}\right) = \frac{6}{10}. \end{aligned}$$

Therefore

$$P(L|E) = \frac{P(E|L)P(L)}{P(E)} = 1 \times \frac{1}{2} \times \frac{10}{6} = \frac{5}{6}.$$

6. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where

$$P(E) = \sum_i P(E|H_i)P(H_i).$$

70% of all women and 40% of all men belong to a particular blood group. Blood has been donated by 20 people of whom 5 are male. If a bottle is picked at random and found not to belong to the group in question, what is the probability that it was donated by a man?

Answer. Let M denote the event of the donor being a man and W the event of the donor being a woman. Let G be the event of the blood belonging to the group and let G^c be the converse. Then we have

$$\begin{aligned} P(M) &= \frac{5}{20}, & P(W) &= \frac{15}{20}, \\ P(G|M) &= \frac{4}{10}, & P(G^c|M) &= \frac{6}{10}, \\ P(G|W) &= \frac{7}{10}, & P(G^c|W) &= \frac{3}{10}. \end{aligned}$$

Therefore

$$\begin{aligned} P(G^c) &= P(G^c|M)P(M) + (G^c|W)P(W) \\ &= \frac{6}{10} \times \frac{5}{20} + \frac{3}{10} \times \frac{15}{20} = \frac{15}{40}, \end{aligned}$$

and

$$\begin{aligned} P(M|G^c) &= \frac{P(G^c|M)P(M)}{P(G^c)} \\ &= \frac{40}{15} \times \frac{6}{10} \times \frac{5}{20} = \frac{2}{5}. \end{aligned}$$

7. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where

$$P(E) = \sum_i P(E|H_i)P(H_i).$$

The Police have found the blood of the jewel thief near the hotel safe. 10% of all women belong to the blood group and 2% of all men. 30% of the hotel staff are women. Assuming that the thief was a member of the hotel staff, what is the probability that the thief was a woman?

8. Let $H_1, H_2, \ldots, H_k$ denote an exhaustive set of mutually exclusive hypotheses representing the possible causes of an event E. Show that

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{P(E)},$$

where $P(E) = \sum_i P(E|H_i)P(H_i)$.

The probability that, on any weekday, the college will receive letters addressed to Dr. A is 1/3. Dr. A, who arrives earlier than any of his colleagues, begins the day by collecting his mail. He has told me that there is a 40% chance that he will attend the college today; and I have noticed that there are no letters in his pigeon hole. In view of there being no mail in his box, what is the probability that he is attending today?

Answer: Let L be the event of my seeing letters in Dr. A's pigeon hole.

Let A be the event that Dr. A's is attending college today.

We are told that

$$P(L|A) = 0, \qquad P(L^c|A) = 1,$$

$$P(A) = \frac{2}{5}, \qquad P(L|A^c) = \frac{1}{3}.$$

We may infer that

$$P(L^c|A^c) = \frac{2}{3}, \qquad P(A^c) = \frac{3}{5}.$$

According to Bayes' Theorem, the probability that Dr. A is in the college, given that his pigeon hole is empty, is

$$P(A|L^c) = \frac{P(L^c|A)P(A)}{P(L^c)}.$$

But

$$\begin{aligned} P(L^c) &= P(L^c|A)P(A) + P(L^c|A^c)P(A^c) \\ &= \left(1 \times \frac{2}{5}\right) + \left(\frac{2}{3} \times \frac{3}{5}\right) = \frac{4}{5}. \end{aligned}$$

Therefore

$$P(A|L^c) = \frac{1 \times 2/5}{4/5} = \frac{2}{5} \times \frac{5}{4} = \frac{1}{2}.$$