EXERCISES IN STATISTICS

Series A, No. 2

1. Let A_1, A_2 be subsets of a sample space S. Show that
$$P(A_1 \cap A_2) \leq P(A_1) \leq P(A_1 \cup A_2) \leq P(A_1) + P(A_2).$$

Answer:

(i) To prove that $P(A_1 \cap A_2) \leq P(A_1)$ we take $P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$ Dividing throughout by $P(A_2|A_1)$ gives
$$P(A_1) = \frac{P(A_1 \cap A_2)}{P(A_2|A_1)} \geq P(A_1 \cap A_2)$$
since $P(A_2|A_1) \leq 1$.

(ii) To prove that $P(A_1) \leq P(A_1 \cup A_2)$, we take $P(A_1 \cup A_2) = P(A_1) + \{P(A_2) - P(A_1 \cap A_2)\}$. But $P(A_2) - P(A_1 \cap A_2) \geq 0$ so
$$P(A_1) \leq P(A_1 \cup A_2).$$

(iii) To prove that $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$ we take $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ and we simply note that $P(A_1 \cap A_2) \geq 0$.

2. Find the probabilities $P(A)$, $P(B)$ when A, B are statistically independent events such that $P(B) = 2P(A)$ and $P(A \cup B) = 5/8$.

Answer: The assumption of independence indicates that $P(A \cap B) = P(A)P(B)$. Using this, and then the fact that $P(B) = 2P(A)$, we find that
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)P(B) \\ &= P(A) + 2P(A) - \{2P(A)\}^2 \\ &= \frac{5}{8}. \end{aligned}$$
Hence
$$2\{P(A)\}^2 - 3P(A) + \frac{5}{8} = 0,$$
$$\Longleftrightarrow$$
$$16\{P(A)\}^2 - 24P(A) + 5 = 0,$$
$$\Longleftrightarrow$$
$$[4P(A) - 1][4P(A) - 5] = 0;$$
and therefore the quadratic equation is solved by $P(A) = 1/4$ and by $P(A) = 5/4$. But we also have $P(A) \leq 1$, so the only viable solution is $P(A) = 1/4$, which implies that $P(B) = 1/2$.

3. The Police have found the blood of the jewel thief near the hotel safe. 10% of all women belong to the blood group and 2% of all men. 30% of the hotel staff are women. Assuming that this was an inside job, what is the probability that the thief was a woman?

Answer: Let M stand for a man and W for a woman. Then, for example, $P(M)$ is the unconditional or "prior" probability that the thief is a man, and $P(B|M)$ is a conditional probability indicating the incidence of the blood group amongst men. We have the following information:

$$P(B|W) = \frac{10}{100}, \qquad P(B|M) = \frac{2}{100},$$
$$P(W) = \frac{30}{100}, \qquad P(M) = \frac{70}{100}.$$

According to Bayes' Theorem, the "posterior"probability that the thief was a woman, given the evidence of the blood, is

$$P(W|B) = \frac{P(B|W)P(W)}{P(B)}.$$

But

$$\begin{aligned} P(B) &= P(B|W)P(W) + P(B|M)P(M) \\ &= \frac{10}{100}.\frac{30}{100} + \frac{2}{100}.\frac{70}{100} \\ &= \frac{44}{1000}. \end{aligned}$$

Therefore

$$\begin{aligned} P(W|B) &= \frac{10}{100}.\frac{30}{100}.\frac{1000}{44} \\ &= \frac{30}{44} \simeq \frac{2}{3}. \end{aligned}$$

4. The probability that, on any weekday, the college will receive letters addressed to Dr. A is 1/3. Dr. A, who arrives earlier than any of his colleagues, begins the day by collecting his mail. He has told me that there is a 40% chance that he will attend the college today; and I have noticed that there are no letters in his pigeon hole. In view of there being no mail in his box, what is the probability that he is attending today?

Answer:
Let L be the event of my seeing letters in Dr. A's pigeon hole.
Let A be the event that Dr. A's is attending college today.
We are told that

$$P(L|A) = 0, \qquad P(L^c|A) = 1,$$

$$P(A) = \frac{2}{5}, \qquad P(L|A^c) = \frac{1}{3}.$$

We may infer that

$$P(L^c|A^c) = \frac{2}{3}, \qquad P(A^c) = \frac{3}{5}.$$

According to Bayes' Theorem, the probability that Dr. A is in the college, given that his pigeon hole is empty, is

$$P(A|L^c) = \frac{P(L^c|A)P(A)}{P(L^c)}.$$

But

$$\begin{aligned} P(L^c) &= P(L^c|A)P(A) + P(L^c|A^c)P(A^c) \\ &= \left(1 \times \frac{2}{5}\right) + \left(\frac{2}{3} \times \frac{3}{5}\right) = \frac{4}{5}. \end{aligned}$$

Therefore

$$P(A|L^c) = \frac{1 \times 2/5}{4/5} = \frac{2}{5} \times \frac{5}{4} = \frac{1}{2}.$$

5. The failure of an electrical circuit is attributable to the failure of either component A of component B or both. The circuit has a probability of failure of 0.4. Component B has a probability of failure of 0.2 Assuming that the probabilities of failure of A and B are independent, what is the probability of failure of A?

Answer: The failure of the circuit is the event $F = A \cup B$. We have $P(F) = P(A \cup B) = 2/5$ and $P(B) = 1/5$. The independence of the components A and B implies that $P(A \cap B) = P(A)P(B)$. It follows that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A)P(B) \\ &= P(A)\{1 - P(B)\} + P(B), \end{aligned}$$

whence

$$\begin{aligned} P(A) &= \Big\{P(A \cup B) - P(B)\Big\}\Big/\Big\{1 - P(B)\Big\} \\ &= P(A)\left\{\frac{2}{5} - \frac{1}{5}\right\}\Big/\left\{1 - \frac{1}{5}\right\} = \frac{1}{5}.\frac{5}{4} = \frac{1}{4}. \end{aligned}$$