Exercises

1. The probability density function governing the minutes of time t spent waiting outside a telephone box is given by $f(t) = ae^{-at}$.

(a) Determine the probability of having to wait for more than t minutes.

(b) Show that the probability of having to wait *more than* $2t$ minutes, given that you have waited for t, is the same as the unconditional probability of having to wait more than t minutes.

What are the implications of this result?

Answer.

(a)The probability of having to wait more than t minutes is given by

$$\begin{aligned} \int_t^\infty f(t)dt &= 1 - \int_0^t f(t)dt \\ &= 1 - \left[-e^{-at} \right]_0^t \\ &= 1 + e^{-at} - 1 = e^{-at}. \end{aligned} \tag{1}$$

(b) The probability of having to wait $2t$ minutes given that you have already waited for t—ie. given that you will have to wait more than t—is

$$\begin{aligned} P(x > 2t | x > t) &= \frac{P(x > 2t)}{P(x > t)} \\ &= \frac{e^{-2at}}{e^{-at}} = e^{-at}. \end{aligned} \tag{2}$$

The implication of this result is that the time spent waiting so far has in no way enhanced the probability that the telephone conversation which is causing the delay will terminate within a given period

It is interesting to relate the waiting-time probabilities to the arrival probabilities which are governed by the Poisson distribution. According to the Poisson distribution, the probability of no arrivals in the interval $(0, t]$ is $f(0, t) = e^{-at}$. This is the same as the probability of having to wait more than t minutes for an arrival—or for a departure in case of the old lady who is occupying the telephone box.

The probability of not having to wait more than t minutes is the probability of an arrival at any point in the interval $(0, t]$. Let $f(t)$ be the probability density function governing the arrivals. Then

$$\begin{aligned} P(x \leq t) &= \int_0^t f(t)dt \\ &= 1 - P(x > t) = 1 - e^{-at}. \end{aligned} \tag{3}$$

This implies

$$f(t) = \frac{d}{dt}\left[1 - 1 - e^{-at}\right] = ae^{-at} \tag{4}$$

This is simply a matter of putting the deduction of (1) into reverse, which is to say that we are now moving from the consequence to the premise.

2. Prove that the function $f(x) = \frac{1}{4}(\frac{3}{4})^x$; $x = 0, 1, 2, \ldots$, constitutes a probability mass function. What is the probability that x will assume any integer value from 0 to 3. Find the value of n such that $P(x < n) = 0.9$ approximately.

Answer.

(i)To prove that $\sum_x f(x) = 1$, we need only use the formula for a geometric progression, Consider the following indefinite sums which are assumed to converge:

$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \ldots, \\ rS &= \quad\ ar + ar^2 + ar^3 + \ldots. \end{aligned} \tag{5}$$

Their difference is

$$S(1 - r) \quad \text{which implies that} \quad S = \frac{a}{1 - r}. \tag{6}$$

In the present case, we have

$$S = \frac{1}{4} \div \left(1 - \frac{3}{4}\right) = 1.$$

(ii) We are required to find

$$P(x = 0, 1, 2, 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3),$$

which is to say that we need to find the sum of the first four terms in a geometric progression. In general, the sum of n terms is found as follows:

$$\begin{aligned} S &= a + ar + \cdots + ar^{n-1} + ar^n + ar^{n+1} + \ldots \\ Sr^n &= \qquad\qquad\qquad\quad ar^n + ar^{n+1} + \ldots \\ S(1-r^n) &= a + ar + \cdots + ar^{n-1} = S_n \end{aligned}$$

Hence, using the expression for S from (7), we get

$$S_n = \frac{a(1-r^n)}{1-r}. \tag{8}$$

In the present case, we have

$$S = \tfrac{1}{4} \times \frac{1-(\frac{3}{4})^4}{1-\frac{3}{4}} = 1 - \left(\tfrac{3}{4}\right)^4 = 0.6825...$$

(ii) We are required to find n such that $P(x < n) = 0.9$. We already know, from the previous result, that $P(x < n) == 1 - (\frac{3}{4})^n$. The solution of

$$0.9 = 1 - \left(\tfrac{3}{4}\right)^n \quad \text{or, equivalently} \quad \left(\tfrac{3}{4}\right)^n = 0.1$$

is

$$n = \frac{\log(0.1}{\log(0.75)} \simeq 8.$$

3. Let x be distributed as $f(x) = e^{-x}$ over the set of positive real numbers. Compute the probability that the random interval $(x, 3x)$ includes the point $x = 3$. What is the expected length of the interval?

Answer. We may reason that the following events are equivalent:

$$\begin{aligned} (x < 3 < 3x) &\Longleftrightarrow (x < 3) \cap (3 < 3x) \\ &\Longleftrightarrow (x < 3) \cap (1 < x) \\ &\Longleftrightarrow (1 < x < 3). \end{aligned}$$

Equally

$$\begin{aligned} (x < 3 < 3x) &\Longleftrightarrow (1 < \frac{3}{x} < 3) \\ &\Longleftrightarrow (1 > \frac{x}{3} > \frac{1}{3}) \\ &\Longleftrightarrow (1 < x < 3). \end{aligned}$$

Then

$$\begin{aligned} P(1 < x < 3) &= \int_1^3 e^{-x} dx \\ &= \big[- e^{-x} \big]_1^3 \\ &= \frac{1}{e} - \frac{1}{e^3} = 0.318... \end{aligned}$$

To compute the length of the interval, we observe that

$$E(x) = \int_0^\infty x e^{-x} dx = 1.$$

The length of the interval is therefore

$$E(3x - x) = E(2x) = 2E(x) = 2.$$