EXERCISES IN MATHEMATICS

Series G, No. 8: Answers

1. Consider the following system of equations:

(i) $$Y_t = C_t + I_t,$$

(ii) $$C_t = \beta Y_{t-1} + A,$$

(iii) $$I_t = \gamma\{C_t - C_{t-1}\}.$$

Demonstrate that these give rise to a 2nd-order difference equation of the form

$$(I + \alpha_1 L + \alpha_2 L^2)Y(t) = A,$$

where $\alpha_1 = -\beta(1+\gamma)$ and $\alpha_2 = \beta\gamma$.

Answer. Substituting from (iii) into (i) gives

$$\begin{aligned} Y_t &= C_t + \gamma\{C_t - C_{t-1}\} \\ &= (1+\gamma)C_t - \gamma C_{t-1}. \end{aligned}$$

Next, by substituting from (ii) into the equation above, we get

$$\begin{aligned} Y_t &= (1+\gamma)\{\beta Y_{t-1} + A\} - \gamma\{\beta Y_{t-2} + A\} \\ &= (1+\gamma)\beta Y_{t-1} - \gamma\beta Y_{t-2} + A. \end{aligned}$$

Rearranging this gives the required result.

2. Use a process of iteration to determine the behaviour of the system of question 1 in the following cases:

(i) $\beta = 0.75$, $\gamma = 0.25$,

(ii) $\beta = 0.6$, $\gamma = 1.5$,

(iii) $\beta = 0.75$, $\gamma = 2.0$.

Start the iteration with the values $Y_1 = 1$ and $Y_{-1} = Y_{-2} = 0$.

Answer. To add some realism to this example, let us vary the terms of the question slightly. Consider taking the first differences of the variables in the equation $(I+\alpha_1 L+\alpha_2 L^2)Y(t) = A(t)$. Here it will be noticed that the constant level A of autonomous expenditure has been replaced by the sequence $A(t)$. On defining

$$y(t) = \nabla Y(t) = Y(t) - Y(t-1) \qquad \text{and} \qquad \delta(t) = \nabla A(t) = A(t) - A(t-1),$$

the equation becomes

$$(I + \alpha_1 L + \alpha_2 L^2)y(t) = \delta(t).$$

Now imagine that autonomous expenditure is constant for an indefinite period prior to time $t = 0$ when it increases by one unit. Thereafter it reverts to its former level. In that case, we may say that $\delta(t) = A(t) - A(t-1)$ contains an isolated unit impulse at $t = 0$; and the sequence may be represented by writing

$$\begin{aligned}\delta(t) &= \{\ldots, a_{-n}, \ldots a_{-1}, a_0, a_1, \ldots, a_n, \ldots\} \\ &= \{\ldots, 0, \ \ldots, 0, 1, 0, \ \ldots, 0, \ \ldots\}.\end{aligned}$$

The corresponding sequence on the output side, which is

$$y(t) = \{\ldots, 0, \ldots 0, y_0, y_1, \ldots, y_n, \ldots\},$$

is described as the impulse response of the system. The impulse response is determined by the following recursion:

$$\begin{aligned}y_0 &= \delta_0 = 1, \\ y_1 &= -a_1\delta_0 = -a_1, \\ y_2 &= -a_1 y_1 - a_2\delta_0 = -a_1 y_1 - a_2, \\ y_3 &= -a_1 y_2 - a_2 y_1, \\ &\vdots \\ y_n &= -a_1 y_{n-1} - a_2 y_{n-2}.\end{aligned}$$

The numbers generated in the three cases are as follows:

t	Case (i)	Case (ii)	Case (iii)
0	1.000	1.000	1.000
1	0.938	1.500	2.250
2	0.691	1.350	3.562
3	0.472	0.675	4.641
4	0.313	−0.203	5.098
5	0.205	−0.911	4.509
6	0.134	−1.185	2.498
7	0.087	−0.957	−1.142
8	0.056	−0.369	−6.317
9	0.036	0.308	−12.500
10	0.024	0.793	−18.650

Here Case (i) represents a sequence which is converging montonically, Case (ii) represents a sequence which is converging in an oscillatory manner, whilst Case (iii) represents a sequence which is diverging in an oscillatory manner.

3. By setting $Y_t = \bar{Y}$, find the analytic expression for the steady-state solution of the system of question 1 and determine its value, when $A = 100$, for those cases which generate a stable solution.

Answer. When $Y(t) = \bar{Y}$, the equation in question (1) becomes

$$(1 + \alpha_1 + \alpha_2)\bar{Y} = A,$$

of which the solution is

$$\bar{Y} = \frac{A}{1 + \alpha_1 + \alpha_2} = \frac{A}{1 - \beta},$$

which is just the level of autonomous expenditure times the Keynesian multiplier. The steady-state can be calculated only in Cases (i) and (ii):

$$\text{Case (i):} \qquad \bar{Y} = \frac{A}{1 - \beta} = 400,$$

$$\text{Case (ii):} \qquad \bar{Y} = \frac{A}{1 - \beta} = 250.$$

4. Find the roots of the equation $\alpha_0 + \alpha_1 z + \alpha_2 z^2 = 0$ for the three cases itemised in question 3.

Answer. We may characterise the behaviour of the solution in terms of the roots λ_1, λ_2 of the primary equation

$$\begin{aligned} 0 &= 1 + \alpha_1 z + \alpha z^2 \\ &= (1 - z/\lambda_1)(1 - z/\lambda_2) \\ &= 1 - (\lambda_1^{-1} + \lambda_2^{-1})z + (\lambda_1\lambda_2)^{-1}z^2. \end{aligned}$$

The results are as follows:

$$\text{Case (i):} \qquad \lambda_1 = 3.45743, \quad \lambda_2 = 1.54257,$$

$$\text{Case (ii):} \qquad \lambda, \lambda^* = 0.83333 \pm i0.64550, \quad |\lambda| = 1.05409,$$

$$\text{Case (iii):} \qquad \lambda, \lambda^* = 0.75000 \pm i0.32275, \quad |\lambda| = 0.81650.$$

In Case (i), the roots are both real-valued which corresponds to the monotonic behaviour of the impulse response; and they are both greater than unity in value which corresponds to the convergent nature of the sequence.

In Case (ii), the roots are conjgate complex numbers, which explains the the oscillatory nature of the impulse response. The modulus of the roots is greater than unity in value which explains the convergent nature of the sequence.

In Case (iii), the roots are conjugate complex numbers, which explains the the oscillatory nature of the impulse response. The modulus of the roots is less than unity in value which explains the divergent nature of the sequence.