EXERCISES IN MATHEMATICS

Series G, No. 7: Answers

1. The following table represents, for each of three categories of employment in a business organisation, indexed by $i = 1, \ldots, 3$, the number of incumbents $x_{i,t-1}$ at the start of a calender year who have survived from the previous year, the number r_{it} recruited throughout the year, the number q_{it} who leave, and the number x_{it} carried over to the next year.

Category	*Incumbents*	*Recruits*	*Leavers*	*Survivors*
1 : Executives	$x_{1,t-1} = 100$	$r_{1t} = 5$	$q_{1t} = 20$	$x_{1t} = 95$
2 : Managers	$x_{2,t-1} = 500$	$r_{2t} = 10$	$q_{2t} = 90$	$x_{2t} = 510$
3 : Supervisors	$x_{3,t-1} = 2000$	$r_{3t} = 1000$	$q_{3t} = 1100$	$x_{3t} = 1800$

Table 1. The overall employment statistics for the organisation in the year t.

The discrepancies in the figures are due to internal promotions. Detailed investigation of the organisation reveals the following pattern of promotion which we may regard as typical for all years:

Promotees	*Executives*	*Managers*	*Supervisors*
1: Executives	80	0	0
2 : Managers	10	400	0
3 : Supervisors	0	100	800

Table 2. The promotion within the organisation during the year t.

By reading along the second row for example, we find that, of the 500 managers on the staff at the start of the year, 400 remained in place, whereas 10 were promoted to the grade of Executive. The remaining 90 left the organisation, as is shown in Table 1.

Assume that, for an indefinite period, the numbers recruited annually to each category are those given in the third column of Table 1. Also assume that, in the ensuing years, the numbers promoted from one category to another and the numbers who leave bear the same proportions to the incumbents as they do in year t. On this basis you are asked

(a) To confirm that the figures in Tables 1 and 2 are mutually consistent,

(b) To forecast the numbers of employees in each category in the next two years,

(c) To describe the long-term tendency of the manpower policy by finding the equilibrium solution for the numbers in each category.

Answer 1: To confirm that the figures in the tables are mutually consistent, we must demonstrate the identity

$$x_{it} = x_{i,t-1} + r_{it} - q_{it} + p_{it} - d_{it},$$

where p_{it} are the numbers promoted to the ith category and d_{it} are the numbers promoted from it. We find that the following equalities hold:

$$\begin{aligned} \text{Executives:}\quad 95 &= 100 + 5 - 20 + 10, \\ x_{1t} &= x_{1,t-1} + r_{1t} - q_{1t} + p_{1t}, \\ \text{Managers:}\quad 510 &= 500 + 10 - 90 + 100 - 10, \\ x_{2t} &= x_{2,t-1} + r_{2t} - q_{2t} + p_{2t} - d_{2t}, \\ \text{Supervisors:}\quad 1800 &= 2000 + 1000 - 1100 - 100, \\ x_{3t} &= x_{3,t-1} + r_{3t} - q_{3t} - d_{3t}. \end{aligned}$$

To forecast the number of employees in future years, we use the equation

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{bmatrix} + \begin{bmatrix} r_{1t} \\ r_{2t} \\ r_{3t} \end{bmatrix}.$$

Here we have

$$a_{11} = \frac{80}{100}, \quad a_{12} = \frac{10}{500}, \quad a_{22} = \frac{400}{500}, \quad a_{23} = \frac{100}{2000}, \quad a_{33} = \frac{800}{2000}.$$

For one period ahead

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \end{bmatrix} = \begin{bmatrix} 91.2 \\ 508 \\ 1720 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.02 & 0 \\ 0 & 0.8 & 0.05 \\ 0 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} 95 \\ 510 \\ 1800 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \\ 1000 \end{bmatrix}.$$

For two periods ahead

$$\begin{bmatrix} x_{1,t+2} \\ x_{2,t+2} \\ x_{3,t+2} \end{bmatrix} = \begin{bmatrix} 88.12 \\ 502.4 \\ 1688 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.02 & 0 \\ 0 & 0.8 & 0.05 \\ 0 & 0 & 0.4 \end{bmatrix} \begin{bmatrix} 91.2 \\ 508 \\ 1720 \end{bmatrix} + \begin{bmatrix} 5 \\ 10 \\ 1000 \end{bmatrix}.$$

To find the equilibrium solution, we may use the method of backsubstitution to solve the system

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & 0 \\ 0 & 1 - a_{22} & -a_{23} \\ 0 & 0 & 1 - a_{33} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \end{bmatrix},$$

where $\bar{x}_i$ is the equilibrium value in the ith category or employment and $\bar{r}_i$ is a constant level of recruitment to the category. We find that

$$\begin{bmatrix} 0.2 & -0.02 & 0 \\ 0 & 0.1 & -0.05 \\ 0 & 0 & 0.6 \end{bmatrix} \begin{bmatrix} 71.\bar{6} \\ 466.\bar{6} \\ 1666.\bar{6} \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 1000 \end{bmatrix}.$$

2. Use the formula

$$y = ((ax + b)x + c)x + d$$

to evaluate the cubic function $y = ax^3 + bx^2 + cx + d$ at the points $x = 0, \ldots, 7$ in the case where

$$\begin{aligned} a &= 1, \\ b &= -6, \\ c &= 1, \\ d &= 12. \end{aligned}$$

Define the difference of the function $y(x)$ by

$$\nabla y(x) = y(x) - y(x-1).$$

(i) Find the first difference $\nabla y(x)$ of the cubic function $y = ax^3 + bx^2 + cx + d$ for the values $x = 1, \ldots, 7$ and find the parameters of the quadratic function which interpolates the corresponding coordinates. Recall that a quadratic function may be interpolated through the coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) using the following expression:

$$\begin{aligned} y(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} &+ y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \\ &+ y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}. \end{aligned}$$

(ii) Find the second difference $\nabla^2 y(x) = \nabla\{\nabla y(x)\}$ of the cubic function for the values $x = 2, \ldots, 7$ and find the parameters of the linear function which interpolates the corresponding coordinates.

(iii) Find the third difference $\nabla^3 y(x) = \nabla\{\nabla^2 y(x)\}$ of the cubic function for the values $x = 3, \ldots, 7$.

Answer 2. The formula $y = ((ax + b)x + c)x + d$, by which the sequence of ordinates is obtained, is known as Horners's method of nested multiplication. Differencing the sequence four times would reduce it to zero. More generally, the sequence of ordinates from a polynomial of degree n would be reduced to zero by differencing $n + 1$ times.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	12	—	—	—
1	8	−4	—	—
2	−2	−10	−6	—
3	−12	−10	0	6
4	−16	−4	6	6
5	−8	8	12	6
6	18	26	18	6
7	68	50	24	6
8	148	80	30	6
9	264	116	36	6
10	422	158	42	6

To find the parameters of the quadratic function which interpolates the coordinates of the once-differenced sequence, we may insert the values $(x_1, y_1) = (1, -4)$, $(x_2, y_2) = (2, -10)$ and $(x_3, y_3) = (3, -10)$ into the formula

$$
\begin{aligned}
y(x) = y_1 &\frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} \\
&+ y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}
\end{aligned}
$$

to give

$$
\begin{aligned}
y(x) &= -4\frac{(x-2)(x-3)}{(1-2)(1-3)} - 10\frac{(x-1)(x-3)}{(2-1)(2-3)} - 10\frac{(x-1)(x-2)}{(3-1)(3-2)} \\
&= -2(x^2 - 5x + 6) + 10(x^2 - 4x + 3) - 5(x^2 - 3x + 2) \\
&= 3x^2 - 15x + 8.
\end{aligned}
$$