EXERCISES IN MATHEMATICS

Series G, No. 6: Answers

Input–Output Analysis.

1. According to the postulate of Leontieff, the value x_{ij} of goods shipped from the ith sector of the economy to the jth sector is proportional to the activity level x_j of the latter: $x_{ij} = a_{ij}x_j$. Also, the activity level of the ith sector is reckoned as the sum of (the values of) the output, x_{ii}, consumed within that sector, the goods, $x_{ij}; j = 1, \ldots, n$, shipped to other sectors, and the goods, y_i, consumed in final demand.

Imagine a closed economy of three sectors which is characterised by the following activity levels and trade flows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}, \quad \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 10 & 30 & 10 \\ 30 & 50 & 20 \\ 10 & 20 & 20 \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 50 \end{bmatrix}.$$

Construct the complete input–output table including a row for the value added to each sector by factor services, and confirm that the various accounting identities have been observed in the construction of the table.

Calculate the matrix $A = [a_{ij}]$ of input–output coefficients. Use the method of Gaussian elimination and the method of back-substitution to solve the equation $(I - A)x = y$ to find the vector $x = [x_1, x_2, x_3]'$ of the activity levels in the three sectors when the levels of final demand are given by $y = [y_1, y_2, y_3]' = [60, 120, 60]'$.

Answer. The trade flows, the activity levels and the final demands are displayed in the following input–output table:

	Sector 1	*Sector 2*	*Sector 3*	*Final Demand*	*Total Demand*
Sector 1	10	30	10	50	100
Sector 2	30	50	20	100	200
Sector 3	10	20	20	50	100
Factors	50	100	50	200	
Activity Level	100	200	100		

The matrix A of input–output coefficients and the Leontieff matrix $I - A$ are

$$A = \begin{bmatrix} 0.1 & 0.15 & 0.1 \\ 0.3 & 0.25 & 0.2 \\ 0.1 & 0.1 & 0.2 \end{bmatrix}, \qquad I - A = \begin{bmatrix} 0.9 & -0.15 & -0.1 \\ -0.3 & 0.75 & -0.2 \\ -0.1 & -0.1 & 0.8 \end{bmatrix}.$$

Imagine that the vector of final demands becomes $y = [y_1, y_2, y_3]' = [60, 120, 60]'$. Then, to find the corresponding activity levels in $x = [x_1, x_2, x_3]'$, we must solve the system $(I - A)x = y$. We have

$$\begin{bmatrix} 0.9 & -0.15 & -0.1 \\ -0.3 & 0.75 & -0.2 \\ -0.1 & -0.1 & 0.8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 120 \\ 60 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 0.9 & -0.15 & -0.1 \\ -0.9 & 2.25 & -0.6 \\ -0.9 & -0.9 & 7.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 360 \\ 540 \end{bmatrix}.$$

Adding the first row to the second row and to the third gives

$$\begin{bmatrix} 0.9 & -0.15 & -0.1 \\ 0.0 & 2.1 & -0.7 \\ 0.0 & -1.05 & 7.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 420 \\ 600 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 0.9 & -0.15 & -0.1 \\ 0.0 & 2.1 & -0.7 \\ 0.0 & -2.1 & 14.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 420 \\ 1200 \end{bmatrix}.$$

Adding the second row of the final expression to the third row gives the following triangular system:

$$\begin{bmatrix} 0.9 & -0.15 & -0.1 \\ 0.0 & 2.1 & -0.7 \\ 0.0 & 0.0 & 13.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 60 \\ 420 \\ 1620 \end{bmatrix}.$$

The solution of this system is

$$x_3 = 120, \qquad x_2 = 240, \qquad x_1 = 120.$$