EXERCISES IN MATHEMATICS

Series G, No. 6: Answers

Input–Output Analysis.

1. According to the postulate of Leontief, the value \( x_{ij} \) of goods shipped from the \( i \)th sector of the economy to the \( j \)th sector is proportional to the activity level \( x_j \) of the latter: \( x_{ij} = a_{ij}x_j \). Also, the activity level of the \( i \)th sector is reckoned as the sum of (the values of) the output, \( x_{ii} \), consumed within that sector, the goods, \( x_{ij}; j = 1, \ldots, n \), shipped to other sectors, and the goods, \( y_i \), consumed in final demand.

Imagine a closed economy of three sectors which is characterised by the following activity levels and trade flows:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  100 \\
  200 \\
  100
\end{bmatrix}, \quad \begin{bmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33}
\end{bmatrix} = \begin{bmatrix}
  10 & 30 & 10 \\
  30 & 50 & 20 \\
  10 & 20 & 20
\end{bmatrix}, \quad \begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} = \begin{bmatrix}
  50 \\
  100 \\
  50
\end{bmatrix}.
\]

Construct the complete input–output table including a row for the value added to each sector by factor services, and confirm that the various accounting identities have been observed in the construction of the table.

Calculate the matrix \( A = [a_{ij}] \) of input–output coefficients. Use the method of Gaussian elimination and the method of back-substitution to solve the equation \((I - A)x = y\) to find the vector \( x = [x_1, x_2, x_3]' \) of the activity levels in the three sectors when the levels of final demand are given by \( y = [y_1, y_2, y_3]' = [60, 120, 60]' \).

Answer. The trade flows, the activity levels and the final demands are displayed in the following input–output table:

<table>
<thead>
<tr>
<th></th>
<th>Sector 1</th>
<th>Sector 2</th>
<th>Sector 3</th>
<th>Final Demand</th>
<th>Total Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sector 1</td>
<td>10</td>
<td>30</td>
<td>10</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>Sector 2</td>
<td>30</td>
<td>50</td>
<td>20</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>Sector 3</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>Factors</td>
<td>50</td>
<td>100</td>
<td>50</td>
<td>200</td>
<td></td>
</tr>
<tr>
<td>Activity Level</td>
<td>100</td>
<td>200</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The matrix \( A \) of input–output coefficients and the Leontief matrix \( I - A \) are
Imagine that the vector of final demands becomes \[ y = [y_1, y_2, y_3]' = [60, 120, 60]' \]. Then, to find the corresponding activity levels in \( x = [x_1, x_2, x_3]' \), we must solve the system \((I - A)x = y\). We have

\[
\begin{bmatrix}
0.9 & -0.15 & -0.1 \\
-0.3 & 0.75 & -0.2 \\
-0.1 & -0.1 & 0.8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
60 \\
120 \\
60
\end{bmatrix}
\iff
\begin{bmatrix}
0.9 & -0.15 & -0.1 \\
-0.9 & 2.25 & -0.6 \\
-0.9 & -0.9 & 7.2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
60 \\
360 \\
540
\end{bmatrix}.
\]

Adding the first row to the second row and to the third gives

\[
\begin{bmatrix}
0.9 & -0.15 & -0.1 \\
0.0 & 2.1 & -0.7 \\
0.0 & -1.05 & 7.1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
60 \\
420 \\
600
\end{bmatrix}
\iff
\begin{bmatrix}
0.9 & -0.15 & -0.1 \\
0.0 & 2.1 & -0.7 \\
0.0 & -2.1 & 14.2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
60 \\
420 \\
1200
\end{bmatrix}.
\]

Adding the second row of the final expression to the third row gives the following triangular system:

\[
\begin{bmatrix}
0.9 & -0.15 & -0.1 \\
0.0 & 2.1 & -0.7 \\
0.0 & 0.0 & 13.5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
60 \\
420 \\
1620
\end{bmatrix}.
\]

The solution of this system is

\[ x_3 = 120, \quad x_2 = 240, \quad x_1 = 120. \]