EXERCISES IN MATHEMATICS

Series G, No. 5: Answers

Matrix Multiplication: Orthonormal Matrices and Rotations.

1. Find following matrix product:

$$\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

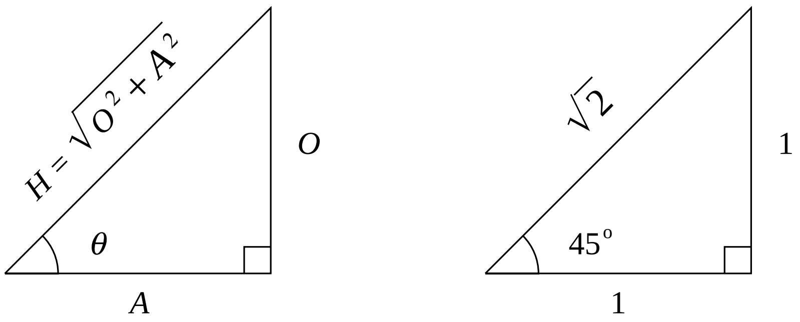
Answer. By two successive matrix multiplications, it is found that

$$\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} S_1 & S_1\rho \\ S_2\rho & S_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} = \begin{bmatrix} S_1^2 & \rho S_1 S_2 \\ \rho S_2 S_1 & S_2^2 \end{bmatrix}.$$

2. Confirm that, if $\theta = 45°$, then

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Answer. By Pythagorus, we have the following triangles:



Hence $\sin\theta = O/H = 1/\sqrt{2}$ and $\cos\theta = A/H = 1/\sqrt{2}$.

3. Use the definitions of a sine and a cosine together with Pythagorus theorem to prove that $\sin^2 + \cos^2 = 1$.

Answer.

$$\sin^2 + \cos^2 = \frac{O^2}{H^2} + \frac{A^2}{H^2} = \frac{O^2 + A^2}{H^2} = 1.$$

4. Confirm that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Answer.

$$\begin{aligned} &\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

A similar result is obtained for the second product.

5. Calculate the product

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1-\rho & 0 \\ 0 & 1+\rho \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Answer.

$$\begin{aligned} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1-\rho & 0 \\ 0 & 1+\rho \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} &= \begin{bmatrix} \frac{1-\rho}{\sqrt{2}} & \frac{1+\rho}{\sqrt{2}} \\ \frac{\rho-1}{\sqrt{2}} & \frac{1+\rho}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \end{aligned}$$

6. Calculate the product

$$\frac{1}{1-\rho^2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Answer.

$$\begin{aligned} &\frac{1}{1-\rho^2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{1-\rho^2} \begin{bmatrix} \frac{1+\rho}{\sqrt{2}} & \frac{1-\rho}{\sqrt{2}} \\ \frac{-(1+\rho)}{\sqrt{2}} & \frac{1-\rho}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}. \end{aligned}$$

7. Show that the product under **6** is the inverse of the product under **5**.

Answer. Using the results of **5** and **6**, it can shown that the product of the two matrices is the identity matrix:

$$\frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This equation may be denoted by $AB = I$. However, there is no need to pursue the evaluations of **5** and **6** in order to obtain this result. Consider writing the equation of **5** as CPC' and that of **6** as CQC', where C' is the matrix of **2** and C is its transpose. Then, according to **4**, we have $C'C = CC' = I$; and, therefore, the product to be evaluated is

$$(CPC')(CQC') = CP(C'C)QC' = CPQC'.$$

But, there is

$$PQ = \frac{1}{1-\rho^2}\begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}\begin{bmatrix} 1-\rho & 0 \\ 0 & 1+\rho \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$

so the product reduces to $CPQC' = CC' = I$, and the result is established anew.

8. Use the method of Gaussian elimination to solve the following set of equations:

$$\begin{bmatrix} 2 & 4 & 3 \\ 2 & 2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ -2 \end{bmatrix}.$$

Answer. On multiplying the third of the equations by 2, the system becomes

$$\begin{bmatrix} 2 & 4 & 3 \\ 2 & 2 & 1 \\ 2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ -4 \end{bmatrix}.$$

On subtracting the first equation from the second and from the third, it is reduced to the following triangular system:

$$\begin{bmatrix} 2 & 4 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \\ -18 \end{bmatrix}$$

of which the three equations are equivalent to

$$\begin{aligned} x_3 &= 2 & \qquad & x_3 = 2, \\ x_2 + x_3 &= 3 & \iff \qquad & x_2 = 3 - x_3 = 1, \\ 2x_1 + 4x_2 + 3x_3 &= 14 & \qquad & x_1 = \tfrac{1}{2}(14 - 4x_2 - 2x_3) = 2. \end{aligned}$$

Therefore $x_3 = 2$, $x_2 = 1$, $x_1 = 2$. These results are verified by substituting them back into the original equation and verifying the equality.