## EXERCISES IN MATHEMATICS

### Series G, No. 4: Answers

**1.** Given that the market rate of interest is 5% per annum, find the market value of

(a) An annuity which promises to pay £250 for the next 10 years,

(b) A perpetuity which promises to pay £100 per annum for ever,

(c) A perpetuity which promises to pay £250 per annum for ever.

**Answer.**

| | | |
|---|---|---|
| Annuity = 250.00 | Annuity = 100.00 | Annuity = 250.00 |
| duration = 10 | duration = $\infty$ | duration = $\infty$ |
| interest = 0.05 | interest = 0.05 | interest = 0.05 |
| discount = 0.9524 | discount = 0.9524 | discount = 0.9524 |
| Value = 1930.4336 | Value = 2000.0 | Value = 5000.0 |

**2.** The following figures represent the yields in £thousands of two investment projects:

| | Year 1 | Year 2 | Year 3 | Year 4 | Year 5 |
|---|---|---|---|---|---|
| Project 1 | 1 | 2 | 4 | 10 | 15 |
| Project 2 | 7 | 10 | 5 | 2 | 1 |

Find the present value of these yields (a) when the prevaling rate of interest is 10% and (b) when the prevailing rate of interest is 15% with a view to determining, in either case, which is the more attractive project.

**Answer.**

**CASE A: r = 10 percent**

| Year | Discount Factor | Discounted Yield of Project 1 | Discounted Yield of Project 2 |
|---|---|---|---|
| 1 | 0.9091 | 0.9091 | 6.3636 |
| 2 | 0.8264 | 1.6529 | 8.2645 |
| 3 | 0.7513 | 3.0053 | 3.7566 |
| 4 | 0.6830 | 6.8301 | 1.3660 |
| 5 | 0.6209 | 9.3138 | 0.6209 |
| TOTALS | | 21.7112 | 20.3716 |

**CASE B: r = 15 percent**

| Year | Discount Factor | Discounted Yield of Project 1 | Discounted Yield of Project 2 |
|---|---|---|---|
| 1 | 0.8696 | 0.8696 | 6.0870 |
| 2 | 0.7561 | 1.5123 | 7.5614 |
| 3 | 0.6575 | 2.6301 | 3.2876 |
| 4 | 0.5718 | 5.7175 | 1.1435 |
| 5 | 0.4972 | 7.4577 | 0.4972 |
| TOTALS | | 18.1871 | 18.5767 |

**3.** Show that, when $L$ increases by a factor of $\gamma = 1 + r$ in the function $Y = \alpha L^\lambda K^\kappa$, the dependent variable $Y$ increases by a factor of $\delta = \gamma^\lambda = (1+r)^\lambda$. Find the value of $\delta$ when $r = 0.2$, $\kappa = 0.4$ and $\lambda = 0.6$ by applying the Taylor-series expansion to the function $\delta(\gamma) = \gamma^\lambda = (1 + r)^\lambda$.

The elasticity of $Y$ in respect of $L$ is commonly described as the % change in $Y$ over the % change in $L$. In the present case, the elasticity is given as $\lambda = 0.6$, and the factor $\gamma = 1.2$ implies a 20% change in $L$. Is it correct to argue that the resulting change in $Y$ will be 60% of 20% which is 12% ?

**Answer.** When the quantity of labour changes by a factor of $\gamma$ the output becomes

$$\delta Y = \alpha(\gamma L)^\lambda K^\kappa = \gamma^\lambda(\alpha L^\lambda K^\kappa) = \gamma^\lambda Y,$$

which implies that $\delta = \gamma^\lambda$. When $\gamma = 1.2$ and $\lambda = 0.6$ we find that $\delta = 1.114$.

The value of $\delta = (1 + r)^\lambda$ may be found via a Taylor-series expansion. Recall that, according to Taylor's theorem,

$$f(x + h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{3!}h^3 f'''(x) + \cdots.$$

By applying the theorem, we find that

$$(1 + r)^\lambda = 1 + \lambda r + \frac{\lambda(\lambda - 1)}{2}r^2 + \frac{\lambda(\lambda - 1)(\lambda - 2)}{3!}r^3 + \cdots.$$

When $\lambda = 0.6$ and $r = 0.2$ we find that

$$\begin{aligned} \lambda r &= \frac{6}{10}.\frac{2}{10} = 0.12 \\ \lambda(\lambda - 1)r^2 &= \frac{6}{10}.\left(\frac{-4}{10}\right).\frac{4}{100} = -0.0096. \end{aligned}$$

Therefore the linear and quadratic approximations to $\delta = (1+r)^\lambda = 1.114$ are respectively

$$1 + \lambda r = 1.12 \qquad \text{and}$$

$$1 + \lambda r + \frac{1}{2}\lambda(\lambda - 1)r^2 = 1.1104.$$

To argue that the change in $Y$ consequent upon the 20% change in $L$ will be 60% of 20% amounts to using a linear approximation. As the % change in $L$ gets larger, the linear approximation becomes less accurate.

**4.** Show that $nC(r-1) + nCr = (n+1)Cr$.

**Answer.** We have

$$(n+1)Cr = \frac{(n+1)!}{(n+1-r)!r!} = \frac{(n+1)n!}{(n+1-r)!r!},$$

$$nC(r-1) = \frac{n!}{(n-r+1)!(r-1)!} = \frac{n!r}{(n-r+1)!r!},$$

$$nCr = \frac{n!}{(n-r)!r!} = \frac{(n-r+1)n!}{(n-r+1)!r!}.$$

Therefore

$$\begin{aligned} nC(r-1) + nCr &= \frac{n!r + n!(n-r+1)}{(n-r+1)!r!} \\ &= \frac{(n+1)n!}{(n-r+1)!r!} = (n+1)Cr. \end{aligned}$$

**5.** Find a simplified expression for the function

$$\begin{aligned} y(x) = y_1\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} &+ y_2\frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} \\ + y_3\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \end{aligned}$$

in the case where $(x_1, y_1) = (0, 3)$, $(x_2, y_2) = (2, 1)$, and $(x_3, y_3) = (4, 3)$.

**Answer.** The substitutions yield

$$\begin{aligned} y(x) &= 3\frac{(x-2)(x-4)}{(0-2)(0-4)} + 1\frac{(x-0)(x-4)}{(2-0)(2-4)} \\ &\quad + 3\frac{(x-0)(x-2)}{(4-0)(4-2)} \\ &= \frac{3}{8}(x^2 - 6x + 8) - \frac{1}{4}(x^2 - 4x) + \frac{3}{8}(x^2 - 2x) \\ &= \frac{1}{2}x^2 - 2x + 3. \end{aligned}$$

This also provides the answer to the first part of Question 6 Exercise 2, which requires that the parameters be found of the quadratic function which interpolates the points $(x, y) = (0, 3), (2, 1), (4, 3)$.

**6.** How many different ways are there of putting the spots on a die?

**Answer:** There are two sets of axes in 3-space: the clockwise and the anti-clockwise. We can imagine fixing the numbers to the vertices of these axes. Therefore the essential question is how many different ways are there of pairing 6 objects.

Let us begin with the permutation of 6 objects : 6! Each permutation implies a set of pairs if we take the ordered elements two-by-two.

Within each permutation the ordering of the pairs does not matter, so we must divide the number of permutation by 3!.

Nor does the order of precedence within the pairs matter, so we must divide again by $2^3$.

The number which we derive is

$$\frac{6!}{3!2^3} = 15.$$

But, if we recall that there are two sets of axes, then we see that the total number of ways of fixing the spots on a die is 30.