EXERCISES IN MATHEMATICS

Series G, No. 3: Answers

1. Let $y = x^n$ where $n = p/q$. By differentiating the equation $y^q = x^p$ on both sides, show that $dy/dx = nx^{n-1}$.

Answer. We have

$$\frac{d}{dy}(y^q) = qy^{q-1} \quad \text{and} \quad \frac{d}{dx}(x^p) = py^{p-1}.$$

Therefore

$$\frac{dy}{dx} = \frac{d(x^p)}{dx}.\frac{dy}{d(y^q)} = \frac{px^{p-1}}{qy^{q-1}} = n\left(\frac{x^p}{y^q}\right)\frac{y}{x} = nx^{n-1},$$

since $x^p/y^q = 1$ and $y/x = x^{n-1}$.

2. The following figures relate to the consumption of natural gas (millions tonnes of coal equivalent) in Britain over a 10-year period:

1966	1967	1968	1969	1970	1971	1972	1973	1974	1975
1.2	2.1	4.8	9.4	17.9	28.8	40.9	44.2	52.9	55.4

(i) Plot a graph of the series and of its logarithm and ascertain whether it follows a process of linear growth or a process of exponential growth.

(ii) Using the formula $y_t = y_0 e^{\rho t}$, calculate the average annual growth (a) for the period 1966–1970 inclusive, (b) for the period 1971–1975 inclusive, (c) for the entire period 1966–1975.

(iii) Calculate the same rates using the formula $y_t = y_0(1 + r)^t$.

(iv) Calculate the linear growth rates using the formula $y_t = y_0 + gt$.

Answer.

(ii) The equation for exponential growth is $y_t = y_0 e^{\rho t}$. The growth rate ρ is given by

$$\rho = \frac{1}{t}\ln\left(\frac{y_t}{y_0}\right).$$

The grow rates over the various periods are calculated as

$$\rho_{66-70} = \frac{1}{5}\ln\left(\frac{179}{12}\right) = 54.05\%,$$

$$\rho_{71-75} = \frac{1}{5}\ln\left(\frac{554}{288}\right) = 13.08\%,$$

$$\rho_{66-75} = \frac{1}{10} \ln \left(\frac{554}{12} \right) = 38.32\%.$$

(iii) The equation for geometric growth is $y_t = y_0(1+r)^t$. This gives

$$\frac{1}{t} \ln \left(\frac{y_t}{y_0} \right) = \ln(1+r) \qquad \text{whence}$$

$$1 + r = \exp \left\{ \frac{1}{t} \ln \left(\frac{y_t}{y_0} \right) \right\}.$$

The grow rates over the various periods are calculated as

$$r_{66-70} = \exp \left\{ \frac{1}{5} \ln \left(\frac{179}{12} \right) \right\} - 1 = 71.69\%,$$

$$r_{71-75} = \exp \left\{ \frac{1}{5} \ln \left(\frac{554}{288} \right) \right\} - 1 = 13.98\%,$$

$$r_{66-75} = \exp \left\{ \frac{1}{10} \ln \left(\frac{554}{12} \right) \right\} - 1 = 46.70\%.$$

3. The costs of a manufacturing firm, as a function of its output q, are given by

$$C = \tfrac{1}{3}q^3 - 5q^2 + 30q + 10.$$

Assume that conditions of perfect competition prevail such that the price $p = \bar{p} = 6$ is not affected by the quantity which the firm brings to the market. Find the output quantity which maximises the firm's profits which are defined by $\pi(q) = R - C$ where $R = p \times q$ is the firm's sales revenue. Confirm that a maximising quantity has been found by evaluating the second derivative $d^2\pi/dq^2$.

Answer. The revenues are $R = 6q$. The profits are given by

$$\pi(q) = R - C = 6q - \tfrac{1}{3}q^3 + 5q^2 - 30q - 10.$$

The first-order condition for a maximum is

$$\frac{d\pi}{dq} = 6 - q^2 + 10q - 30 = 0,$$

which is rearranged to give

$$\begin{aligned} 0 &= q^2 - 10q + 24 \\ &= (q-6)(q-4). \end{aligned}$$

There are two solutions: $q = 4, 6$. To determine their status, we must evaluate the second derivative at either point:

$$\frac{d^2\pi}{dq^2} = -2q + 10.$$

At $q = 4$ the second derivative is positive which indicates a minimum. At $q = 6$ it is negative which indicates a maximum.

4. Find the values of x which satisfy the condition $dy(x)/dx = 0$ in each of the following cases, and ascertain whether they correspond to maxima, to minima or to points of inflection:

(i) $y = \frac{1}{3}x^3 + x^2 + x$,

(ii) $y = \frac{1}{3}x^3 - x + 10$,

(iii) $y = \dfrac{x^3 + 4x^2 + 5x + 2}{x + 1}$,

(iv) $y = x^3 + 2x^2 - 7x + 1$,

(v) $y = (x^2 - 1)^2$,

(vi) $y = \dfrac{1 + x}{x^2}$.

Answer.

(i) We have

$$f(x) = \tfrac{1}{3}x^3 + x^2 + x, \qquad f''(x) = 2x + 2,$$
$$f'(x) = x^2 + 2x + 1, \qquad f'''(x) = 2.$$

The first-order condition is $f'(x) = x^2 + 2x + 1 = (x + 1)^2 = 0$, which implies a unique solution of $x = -1$. Then $f''(-1) = 0$ and $f'''(-1) = 2$. This indicates a point of inflection at $x = -1$. Also $f'(x) > 0$ for $x < -1$ and for $x > -1$; and so $f(x)$ is a non-decreasing function of x.

(ii) We have

$$f(x) = \tfrac{1}{3}x^3 - x + 10, \qquad f''(x) = 2x,$$
$$f'(x) = x^2 - 1, \qquad f'''(x) = 2.$$

The first-order condition is $f'(x) = x^2 - 1 = 0$ which has the solutions $x = \pm 1$. Then $f''(1) = 2$ and $f''(-1) = -2$. This indicates a minimum $x = 1$ and a maximum at $x = -1$.

(iii) Both the numerator and denominator contain the factor $x + 1$ and we have

$$f(x) = x^2 + 3x + 2, \qquad f''(x) = 2,$$
$$f'(x) = 2x + 3, \qquad f'''(x) = 0.$$

The first-order condition is $f'(x) = 2x + 3 = 0$ which implies a unique solution of $x = -3/2$ which is a minimum.

(iv) We have

$$f(x) = x^3 + 2x^2 - 7x + 1, \qquad f''(x) = 6x + 4,$$
$$f'(x) = 3x^2 + 4x - 7, \qquad f'''(x) = 6.$$

The first-order condition is $f'(x) = 3x^2 + 4x - 7 = (x-1)(3x+7) = 0$ which indicates solutions of $x = 1, -2\frac{1}{3}$. The second derivatives at these points are $f''(1) = 10$ and $f''(-2\frac{1}{3}) = -10$ which indicates that $x = 1$ gives a minimum and that $x = -2\frac{1}{3}$ gives a maximum.

(v) We have

$$f(x) = (x^2 - 1)^2, \qquad f''(x) = 12x^2 - 4,$$
$$f'(x) = 4x(x^2 - 1) = 4x^3 - 4x, \qquad f'''(x) = 24x.$$

The first-order condition is $f'(x) = x(4x^2 - 4) = 0$ which indicates solutions of $x = \pm 1$ and $x = 0$. The second derivatives at these points are $f''(\pm 1) = 0$ and $f''(0) = -4$, which indicates minima at $x = \pm 1$ and a maximum at $x = 0$.

(vi) We have

$$f(x) = x^{-2} + x^{-1}, \qquad f''(x) = 6x^{-4} + 2x^{-3},$$
$$f'(x) = -2x^{-3} - x^{-2}, \qquad f'''(x) = -24x^{-5} - 6x^{-4}.$$

The first-order condition is $f'(x) = -2x^{-3} - x^{-2} = 0$ which entails the condition $2 + x = 0$. of which the solution is $x = -2$. The second derivative at this point is $f''(-2) = 2/16$, which indicates a minimum.

5. Let $Y = \alpha L^\lambda K^\kappa$. Show that

$$\frac{\partial Y}{\partial L}\frac{L}{Y} = \lambda \quad \text{and} \quad \frac{\partial Y}{\partial K}\frac{K}{Y} = \kappa.$$

Answer. The derivatives are

$$\frac{\partial Y}{\partial L}.\frac{L}{Y} = \alpha\{\lambda L^{\lambda-1}\}K^\kappa.\frac{L}{Y} = \lambda\frac{\{\alpha L^\lambda K^\kappa\}}{Y} = \lambda,$$
$$\frac{\partial Y}{\partial K}.\frac{L}{K} = \alpha L^\lambda\{\kappa K^{\kappa-1}\}.\frac{K}{Y} = \kappa\frac{\{\alpha L^\lambda K^\kappa\}}{Y} = \kappa.$$

The function in question is the Cobb–Douglas production function which gives the total output Y of an enterprise in terms of the quantities of labour L and capital K. The returns to scale are indicated by the sum of the exponents:

$$\lambda + \kappa < 1 \Longrightarrow \text{decreasing returns to scale},$$
$$\lambda + \kappa = 1 \Longrightarrow \text{constant returns to scale},$$
$$\lambda + \kappa > 1 \Longrightarrow \text{economies of scale}.$$