EXERCISES IN MATHEMATICS

Series G, No. 2: Answers

1. Let $y = uvw$, where $u = u(x)$, $v = v(x)$ and $w = w(x)$ are each a function of x. Establish a product rule which should enable you to find the derivative dy/dx.

Answer. Define $p = vw$ so that $y = up$. Then

$$\frac{dp}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx},$$

and

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dp}{dx} + p\frac{du}{dx}\\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.\end{aligned}$$

More generally, if

$$y = a_1 \times a_2 \times \cdots \times a_n = \prod_{j=1}^{n} a_j,$$

where $a_j = a_j(x); j = 1, \ldots, n$ are each a function of x, then

$$\frac{dy}{dx} = \sum_{j=1}^{n} \frac{y}{a_j}\frac{da_j}{dx}.$$

Observe that this result may be used to establish the rule that

$$\frac{d}{dx}x^n = nx^{n-1},$$

which we might otherwise establish from first principles using the binomial theorem.

2. Over a 20-year period, the number of households of Casterfield has grown from 15,000 to 22,000. The average number of cars per household has increased from 0.5 to 0.85, and the average daily distance travelled by private cars has increased from 12 miles to 15.5 miles.

Imagine that these quantities are governed by equations in the form of $y_t = y_0 e^{rt}$, where y_0 and y_t are respectively the quantity at the beginning and the end of the period, e is the natural number and r is the growth rate. Equivalently, $\ln y_t = \ln y_0 + rt$, where ln denotes a natural logarithm.

(a) Find the annual growth rate of the number of households, of car ownership and of average daily mileage separately. (b) Find the annual growth rate for the overall daily mileage travelled by the citizens of Casterfield.

Answer. The equation which governs the growth of y indicates that

$$r = \frac{1}{t}\ln\left(\frac{y_t}{y_0}\right) = \frac{\ln y_t - \ln y_0}{t}.$$

Let N, C and M denote the number of household, the cars per household and the average daily mileage respectively. The growth rates per annum are

$$r_N = \frac{1}{t}\ln\left(\frac{N_t}{N_0}\right) = \frac{1}{20}\ln\left(\frac{22,000}{15,000}\right) = 1.91496\%,$$

$$r_C = \frac{1}{t}\ln\left(\frac{C_t}{C_0}\right) = \frac{1}{20}\ln\left(\frac{0.85}{0.50}\right) = 2.65314\%,$$

$$r_M = \frac{1}{t}\ln\left(\frac{M_t}{M_0}\right) = \frac{1}{20}\ln\left(\frac{15.5}{12.0}\right) = 1.27967\%,$$

whence

$$r = r_N + r_C + r_M = 5.84777\%$$

is the annual growth rate of overall daily mileage travelled by the citizens. Alternatively, we can calculate

$$r = \frac{1}{t}\ln\left(\frac{N_t C_t M_t}{N_0 C_0 M_0}\right) = \frac{1}{20}\ln\left(\frac{22 \times 85 \times 155}{15 \times 50 \times 120}\right) = 5.84777\%.$$

3. Use two iterations of the formula

$$\xi_{r+1} = \frac{1}{2}\left(\xi_r + \frac{N}{\xi_r}\right)$$

to find a approximation to $\sqrt{13}$ by setting $N = 13$ and using $\xi_0 = 4$ as a starting value, and check the result by finding ξ^2.

Answer. With $\xi_0 = 4$ and $N = 13$, the formula gives

$$\xi_1 = \frac{1}{2}\left(4 + \frac{13}{4}\right) = \frac{29}{8} = 3.625 \qquad \text{and} \qquad \xi_1^2 = 13.140625.$$

Then with $\xi_1 = 29/8$ the formula gives

$$\xi_2 = \frac{1}{2}\left(\frac{29}{8} + \frac{13 \times 8}{29}\right) = \frac{841 + 832}{464} = 3.60560 \qquad \text{and} \qquad \xi_2^2 = 13.00038.$$

4. Find an approximate solution to the equation

$$f(x) = 2x^5 - 2x - 190 = 0$$

via three iterations of Newton's formula

$$x_{r+1} = x_r - \left\{f'(x_r)\right\}^{-1} f(x_r),$$

using $x_0 = 2$ as the starting value.

Answer. If fact, it is instructive to take two more iterations in order to establish the convergence of the algorithm. With $x_0 = 2$, we get

$$\begin{aligned} x_1 &= 2.82278490 & x_1^2 &= 7.96811460 \\ x_2 &= 2.56556535 & x_2^2 &= 6.58212555 \\ x_3 &= 2.50255704 & x_3^2 &= 6.26279173 \\ x_4 &= 2.49920440 & x_4^2 &= 6.24602262 \\ x_5 &= 2.49919534 & x_5^2 &= 6.24597733. \end{aligned}$$

5. Find the equation of the straight line ℓ which passes through the points $(x, y) = (3, 13), (7, 25)$. Find the equation of the straight line perpendicular to ℓ which bisects ℓ midway between these points.

Answer. Denote the equation by $y = \alpha + \beta x$ and the pair of points by $(x_1, y_1), (x_2, y_2)$. Then

$$\begin{aligned} y_2 &= \alpha + \beta x_2 \\ y_1 &= \alpha + \beta x_1 \\ \hline y_2 - y_1 &= \beta(x_2 - x_1), \end{aligned}$$

so

$$\beta = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{and} \qquad \alpha = y_1 - \beta x_1.$$

In this case, we have $\beta = 3$, $\alpha = 4$; and therefore the equation of the line ℓ is

$$y = 3x + 4.$$

Let the equation of the line perpendicular to ℓ be

$$y = \gamma + \delta x.$$

The slope of this line is $\delta = -\beta^{-1} = -\frac{1}{3}$ The line passes through the mid-point on the chord between $(3, 13), (7, 25)$, which is the point $(x_3, y_3) = (5, 19)$. Hence $y_3 = \delta x_3 + \gamma$ implies $\gamma = y_3 - \delta x_3 = 19 + 5/3 = 20\frac{2}{3}$.

6. Find the parameters a, b and c of the quadratic $q(x) = ax^2 + bx + c$ with the coordinates $(x, y) = (0, 3), (2, 1), (4, 3)$ and of the line $\ell(x)$ which passes through the points $(x, y) = (0, 3), (4, 3)$. Find the equation for $\phi(x) = q(x) - \ell(x)$, and show that $\phi'(x) = 0$ for some value $x \in (0, 4)$.

Answer. The parameters of the quadratic may be found by found solving the following three equations obtained by substituting the values from the coordinates (x, y) into the equation $y = ax^2 + bx + c$:

$$\begin{aligned} 3 &= c, \\ 1 &= 4a + 2b + c, \\ 3 &= 16a + 4b + c. \end{aligned}$$

The coefficient are $a = \frac{1}{2}, b = -2, c = 3$ so the quadratic equation is

$$q(x) = \frac{1}{2}x^2 - 2x + 3.$$

The equation of the line passing through the points $(0, 3), (4, 3)$ is $\ell(x) = 3$, whence

$$\begin{aligned} \phi(x) &= q(x) - \ell(x) \\ &= \frac{1}{2}x^2 - 2x \end{aligned}$$

and

$$\phi'(x) = x - 2.$$

Setting $\phi'(x) = 0$, which the first-order condition for a minimum, and solving the resulting equation gives $x = 2$.