EXERCISES IN MATHEMATICS

Series F, No. 6

1. Let the growth of the fish population be governed by the equation

$$\frac{dy}{dt} = \rho y \left(\frac{\gamma - y}{\gamma} \right),$$

and imagine that the rate of extraction is constant at the level of h. Find an explicit solution to the quadratic equation

$$\frac{dy}{dt} - h = 0,$$

which is when the fish manage to maintain their numbers in the face of harvesting. Discuss the implication of the following three conditions:

(a) $h > \gamma\rho/4$,

(b) $h = \gamma\rho/4$,

(c) $h < \gamma\rho/4$.

Answer. When the fish stocks manage to regenerate themselves, the rate of growth of their population is equal to the rate at which they are extracted from the sea. In that case,

$$\begin{aligned}\frac{dy}{dt} - h &= \rho y \left(\frac{\gamma - y}{\gamma} \right) - h \\ &= \rho y - \frac{\rho y^2}{\gamma} - h = 0.\end{aligned}$$

Therefore, to find the size of the population, we must solve the quadratic equation

$$\rho y^2 - \rho\gamma y + h\gamma = 0.$$

The solution is

$$y = \frac{\rho\gamma \pm \sqrt{\rho^2\gamma^2 - 4\rho\gamma h}}{2\rho}.$$

The nature of the solution depends upon the value of the discriminant.

(a) If $\rho^2\gamma^2 - 4\rho\gamma h > 0$, then there are two solutions y_L and y_U with $y_L < y_U$. The lower population size y_L is the result of over-intensive fishing. The equilibrium between the rate of extraction h and the rate of regeneration dy/dt is unstable. Any increase in the rate of extraction would drive the fish population to extinction. The higher population size y_U corresponds to a case where the harvest h is extracted from a large population of fish whose numbers have not been greatly diminished by the activities of fishermen.

(b) If $\rho^2\gamma^2 - 4\rho\gamma h = 0$, then there is a unique solution for y. In that case, the rate of harvesting h is at a maximum. Any increase in the rate would drive the fish population to extinction.

(c) If $\rho^2\gamma^2 - 4\rho\gamma h < 0$, then there is no real-valued solution and the rate of harvesting is unsustainable. It threatens to drive the fish population to extinction.

2. Find the first-order and second-order partial derivatives of the following functions:

(i) $z = 3x^2 + 4xy + y^2$,

(ii) $z = x^3 + x^2y + 2y^2x + 2y^3$,

(iii) $z = ax/y^2$,

(iv) $z = 1/xy$.

Answer.

(i)
$$\begin{aligned} f &= 3x^2 + 4xy + y^2, \\ f_x &= 6x + 4y, \\ f_y &= 4x + 2y, \\ f_{xx} &= 6, \\ f_{yy} &= 2, \\ f_{xy} &= f_{yx} = 4. \end{aligned}$$

(ii)
$$\begin{aligned} f &= x^3 + x^2y + 2y^2x + 2y^3, \\ f_x &= 3x^2 + 2xy + 2y^2, \\ f_y &= x^2 + 4xy + 6y^2, \\ f_{xx} &= 6x + 2y, \\ f_{yy} &= 4x + 12y, \\ f_{xy} &= f_{yx} = 2x + 4y. \end{aligned}$$

(iii)
$$\begin{aligned} f &= ax/y^2, \\ f_x &= a/y^2, \\ f_y &= -2ax/y^3, \\ f_{xx} &= 0, \\ f_{yy} &= 6ax/y^4, \\ f_{xy} &= f_{yx} = -2a/y^3. \end{aligned}$$

(iv)

$$
\begin{aligned}
f &= 1/xy, \\
f_x &= -1/(x^2y), \\
f_y &= -1/(y^2x), \\
f_{xx} &= 2/(x^3y), \\
f_{yy} &= 2/(y^3x), \\
f_{xy} &= f_{yx} = 2/(x^2y^2).
\end{aligned}
$$

3. Find the extreme values of each of the following three functions and determine whether they are maxima or minima:

(i) $z = x^2 + xy + 2y^2 + 3$,

(ii) $z = -x^2 + xy - y^2 + 2x + y$,

(iii) $z = e^{2x} - 2x + 2y^2 + 3$.

Answer.

(i)

$$
\begin{aligned}
f &= x^2 + xy + 2y^2 + 3, \\
f_x &= 2x + y, \\
f_y &= x + 4y, \\
f_{xx} &= 2, \\
f_{yy} &= 4, \\
f_{xy} &= f_{yx} = 1 \\
f_{xx}f_{yy} - f_{xy}^2 &= 7.
\end{aligned}
$$

From the solution of first-order conditions $f_x = 0, f_y = 0$, we find that the stationary point is $(x, y) = (0, 0)$. At that point, we find that $f(x, y) = 3$. Since $f_{xx}, f_{yy}, f_{xx}f_{yy} - f_{xy}^2 > 0$, this is a minimum.

(ii)

$$
\begin{aligned}
f &= -x^2 + xy - y^2 + 2x + y, \\
f_x &= -2x + y + 2, \\
f_y &= x - 2y + 1, \\
f_{xx} &= -2, \\
f_{yy} &= 1, \\
f_{xy} &= f_{yx} = 1 \\
f_{xx}f_{yy} - f_{xy}^2 &= -3
\end{aligned}
$$

From the solution of first-order conditions $f_x = 0, f_y = 0$, we find that the stationary point is $(x, y) = (5/3, 4/3)$. This is neither a maximum nor a minimum point, since the condition $f_{xx}f_{yy} - f_{xy}^2 > 0$ is violated.

(iii)
$$\begin{aligned} f &= e^{2x} - 2x + 2y^2 + 3,\\ f_x &= 2e^{2x} - 2,\\ f_y &= 4y,\\ f_{xx} &= 4e^{2x},\\ f_{yy} &= 4,\\ f_{xy} &= f_{yx} = 0\\ f_{xx}f_{yy} - f_{xy}^2 &= 16e^{2x} \end{aligned}$$

The first-order condition $f_x = 0$ indicates that $2x = 0$, whence $x = 0$. The condition $f_y = 0$ indicates that $y = 0$. At the stationary point $(x, y) = (0, 0)$, we find that $f(x, y) = 4$. Also at that point, $f_{xx} = f_{yy} = 4$ and $f_{xx}f_{yy} - f_{xy}^2 = 16$, which together indicate a minimum. Note that it is unnecessary to evaluate the final condition, since $f(x, y)$ is a sum of a functions in x and y alone which may be minimised separately.

4. Use the Lagrangean multiplier technique to find the maximum of the function

$$f(x, y) = 20x^{1/2}y^{1/2}$$

subject to the constraint

$$g(x, y) = 300 - 2x - 5y.$$

Answer. The Lagrangean function is

$$\begin{aligned} L(x, y, \lambda) &= f(x, y) + \lambda g(x, y)\\ &= 20x^{1/2}y^{1/2} + \lambda(30 - 2x - 5y). \end{aligned}$$

The first-order conditions for optimisation are obtained by setting the partial derivatives to zero:

(i) $$\frac{\partial L}{\partial x} = 10x^{-1/2}y^{1/2} - 2\lambda = 0,$$

(ii) $$\frac{\partial L}{\partial y} = 10y^{-1/2}x^{1/2} - 5\lambda = 0,$$

(iii) $$\frac{\partial L}{\partial \lambda} = 30 - 2x - 5y = 0.$$

By forming the ration of (i) and (ii), we get

(iv) $$\frac{y}{x} = \frac{2}{5} \qquad \text{or} \qquad 2x = 5y.$$

Then, substituting to x in (ii) gives

(v) $$30 - 10y = 0 \qquad \text{or} \qquad y = 3.$$

It follows from (iv) that

(vi) $$x = 7\tfrac{1}{2}.$$