EXERCISES IN MATHEMATICS

Series F, No. 3: Answers

1. Let $y = uvw$, where $u = u(x)$, $v = v(x)$ and $w = w(x)$ are each a function of x. Establish a product rule which should enable you to find the derivative dy/dx.

Answer. Define $p = vw$ so that $y = up$. Then

$$\frac{dp}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx},$$

and

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dp}{dx} + p\frac{du}{dx}\\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}.\end{aligned}$$

More generally, if

$$y = a_1 \times a_2 \times \cdots \times a_n = \prod_{j=1}^{n} a_j,$$

where $a_j = a_j(x); j = 1, \ldots, n$ are each a function of x, then

$$\frac{dy}{dx} = \sum_{j=1}^{n} \frac{y}{a_j}\frac{da_j}{dx}.$$

Observe that this result may be used to establish the rule that

$$\frac{d}{dx}x^n = nx^{n-1},$$

which we might otherwise establish from first principles using the binomial theorem.

2. Over a 20-year period from 1965 to 1984, an index of the volume of U.K exports has grown from 39.7 to 94.7. The index of unit value has increased from 14.6 to 95.0.

Imagine that these quantities are governed by equations in the form of $y_t = y_0 e^{rt}$, where y_0 and y_t are respectively the quantity at the beginning and the end of the period, e is the natural number and r is the growth rate.

Find the annual growth rate for the volume and the value indices and for their product.

Answer. Taking natural logarithms of the equation $y_t = y_0 e^{rt}$ gives $\ln y_t = \ln y_0 + rt$, which indicates that

$$r = \frac{1}{t}\ln\left(\frac{y_t}{y_0}\right) = \frac{\ln y_t - \ln y_0}{t}.$$

Let M, L denote the volume index and the value index respectively. Then growth the rates per annum are

$$r_M = \frac{1}{t}\ln\left(\frac{M_t}{M_0}\right) = \frac{1}{20}\ln\left(\frac{94.7}{39.7}\right) = 4.34681\%,$$

$$r_L = \frac{1}{t}\ln\left(\frac{L_t}{L_0}\right) = \frac{1}{20}\ln\left(\frac{95.0}{14.6}\right) = 9.36428\%,$$

whence the overall growth of trade is given by

$$r = r_M + r_L = 13.71109\%.$$

3. The costs of a manufacturing firm, as a function of its output q, are given by

$$C = \tfrac{1}{3}q^3 - 5q^2 + 30q + 10.$$

Assume that conditions of perfect competition prevail such that the price $p = 6$ is not affected by the quantity which the firm brings to the market. Find the output quantity which maximises the firm's profits which are defined by $\pi(q) = R - C$ where $R = p \times q$ is the firm's sales revenue. Confirm that a maximising quantity has been found by evaluating the second derivative $d^2\pi/dq^2$.

Answer. The revenues are $R = 6q$. The profits are given by

$$\pi(q) = R - C = 6q - \tfrac{1}{3}q^3 + 5q^2 - 30q - 10.$$

The first-order condition for a maximum is

$$\frac{d\pi}{dq} = 6 - q^2 + 10q - 30 = 0,$$

which is rearranged to give

$$\begin{aligned} 0 &= q^2 - 10q + 24 \\ &= (q-6)(q-4). \end{aligned}$$

There are two solutions: $q = 4, 6$. To determine their status, we must evaluate the second derivative at either point:

$$\frac{d^2\pi}{dq^2} = -2q + 10.$$

At $q = 4$, the second derivative is positive which indicates a minimum. At $q = 6$, it is negative which indicates a maximum.

4. Find the values of x which satisfy the condition $dy(x)/dx = 0$ in each of the following cases, and ascertain whether they correspond to maxima, to minima or to points of inflection:

(i) $y = \frac{1}{3}x^3 + x^2 + x$,

(ii) $y = \frac{1}{3}x^3 - x + 10$,

(iii) $y = \dfrac{x^3 + 4x^2 + 5x + 2}{x+1}$,

(iv) $y = x^3 + 2x^2 - 7x + 1$,

(v) $y = (x^2 - 1)^2$,

(vi) $y = \dfrac{1+x}{x^2}$.

Answer.

(i) We have

$$f(x) = \tfrac{1}{3}x^3 + x^2 + x, \qquad f''(x) = 2x + 2,$$
$$f'(x) = x^2 + 2x + 1, \qquad f'''(x) = 2.$$

The first-order condition is $f'(x) = x^2 + 2x + 1 = (x+1)^2 = 0$, which implies a unique solution of $x = -1$. Then $f''(-1) = 0$ and $f'''(-1) = 2$. This indicates a point of inflection at $x = -1$. Also $f'(x) > 0$ for $x < -1$ and for $x > -1$; and so $f(x)$ is a non-decreasing function of x.

(ii) We have

$$f(x) = \tfrac{1}{3}x^3 - x + 10, \qquad f''(x) = 2x,$$
$$f'(x) = x^2 - 1, \qquad f'''(x) = 2.$$

The first-order condition is $f'(x) = x^2 - 1 = 0$ which has the solutions $x = \pm 1$. Then $f''(1) = 2$ and $f''(-1) = -2$. This indicates a minimum $x = 1$ and a maximum at $x = -1$.

(iii) Both the numerator and denominator contain the factor $x + 1$ and we have

$$f(x) = x^2 + 3x + 2, \qquad f''(x) = 2,$$
$$f'(x) = 2x + 3, \qquad f'''(x) = 0.$$

The first-order condition is $f'(x) = 2x + 3 = 0$ which implies a unique solution of $x = -3/2$ which is a minimum.

(iv) We have

$$f(x) = x^3 + 2x^2 - 7x + 1, \qquad f''(x) = 6x + 4,$$
$$f'(x) = 3x^2 + 4x - 7, \qquad f'''(x) = 6.$$

The first-order condition is $f'(x) = 3x^2 + 4x - 7 = (x-1)(3x+7) = 0$ which indicates solutions of $x = 1, -2\frac{1}{3}$. The second derivatives at these points are $f''(1) = 10$ and $f''(-2\frac{1}{3}) = -10$ which indicates that $x = 1$ gives a minimum and that $x = -2\frac{1}{3}$ gives a maximum.

(v) We have

$$f(x) = (x^2 - 1)^2, \qquad f''(x) = 12x^2 - 4,$$
$$f'(x) = 4x(x^2 - 1) = 4x^3 - 4x, \qquad f'''(x) = 24x.$$

The first-order condition is $f'(x) = x(4x^2 - 4) = 0$ which indicates solutions of $x = \pm 1$ and $x = 0$. The second derivatives at these points are $f''(\pm 1) = 0$ and $f''(0) = -4$, which indicates minima at $x = \pm 1$ and a maximum at $x = 0$.

(vi) We have

$$f(x) = x^{-2} + x^{-1}, \qquad f''(x) = 6x^{-4} + 2x^{-3},$$
$$f'(x) = -2x^{-3} - x^{-2}, \qquad f'''(x) = -24x^{-5} - 6x^{-4}.$$

The first-order condition is $f'(x) = -2x^{-3} - x^{-2} = 0$ which entails the condition $2 + x = 0$. of which the solution is $x = -2$. The second derivative at this point is $f''(-2) = 2/16$, which indicates a minimum.

5. Find the equation of the straight line ℓ which passes through the points $(x, y) = (3, 13), (7, 25)$. Find the equation of the straight line perpendicular to ℓ which bisects ℓ midway between these points.

Answer. Denote the equation by $y = \alpha + \beta x$ and the pair of points by $(x_1, y_1), (x_2, y_2)$. Then

$$\begin{aligned} y_2 &= \alpha + \beta x_2 \\ y_1 &= \alpha + \beta x_1 \\ \hline y_2 - y_1 &= \beta(x_2 - x_1), \end{aligned}$$

so

$$\beta = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{and} \qquad \alpha = y_1 - \beta x_1.$$

In this case, we have $\beta = 3$, $\alpha = 4$; and therefore the equation of the line ℓ is

$$y = 3x + 4.$$

Let the equation of the line perpendicular to ℓ be

$$y = \gamma + \delta x.$$

The slope of this line is $\delta = -\beta^{-1} = -\frac{1}{3}$ The line passes through the mid-point on the chord between $(3, 13), (7, 25)$, which is the point $(x_3, y_3) = (5, 19)$. Hence $y_3 = \delta x_3 + \gamma$ implies $\gamma = y_3 - \delta x_3 = 19 + 5/3 = 20\frac{2}{3}$.

6. Find the parameters a, b and c of the quadratic $q(x) = ax^2 + bx + c$ with the coordinates $(x, y) = (0, 3), (2, 1), (4, 3)$ and of the line $\ell(x)$ which passes through the points $(x, y) = (0, 3), (4, 3)$. Find the equation for $\phi(x) = q(x) - \ell(x)$, and show that $\phi'(x) = 0$ for some value $x \in (0, 4)$.

Answer. The parameters of the quadratic may be found by found solving the following three equations obtained by substituting the values from the coordinates (x, y) into the equation $y = ax^2 + bx + c$:

$$\begin{aligned} 3 &= c, \\ 1 &= 4a + 2b + c, \\ 3 &= 16a + 4b + c. \end{aligned}$$

The coefficient are $a = \frac{1}{2}, b = -2, c = 3$ so the quadratic equation is

$$q(x) = \frac{1}{2}x^2 - 2x + 3.$$

The equation of the line passing through the points $(0, 3), (4, 3)$ is $\ell(x) = 3$, whence

$$\begin{aligned} \phi(x) &= q(x) - \ell(x) \\ &= \frac{1}{2}x^2 - 2x \end{aligned}$$

and

$$\phi'(x) = x - 2.$$

Setting $\phi'(x) = 0$, which the first-order condition for a minimum, and solving the resulting equation gives $x = 2$.