EXERCISES IN MATHEMATICS

Series F, No. 2: Answers

First Principles

1. Differentiate from first principles $y = x^2 - 4x$.

Answer. We have $y + \delta y = (x + \delta x)^2 - 4(x + \delta x)$. Subtracting $y = x^2 - 4x$ gives

$$\begin{aligned} \delta y &= \big[(x+\delta x)^2 - 4(x+\delta x)\big] - [x^2 - 4x] \\ &= x^2 + 2x(\delta x) + (\delta x)^2 - 4x - 4(\delta x) - x^2 + 4x \\ &= 2x(\delta x) - 4(\delta x) + (\delta x)^2. \end{aligned}$$

Dividing by δx gives

$$\frac{\delta y}{\delta x} = 2x - 4 + \delta x;$$

and the limit as $\delta x \to 0$ is

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) \\ &= 2x - 4. \end{aligned}$$

2. Differentiate from first principles $f(x) = 1/x$.

Answer. Subtracting $y = 1/x$ from $y + \delta y = 1/(x + \delta x)$ gives

$$\begin{aligned} \delta y = \frac{1}{x+\delta x} - \frac{1}{x} &= \frac{x - (x+\delta x)}{(x+\delta x)x} \\ &= \frac{-\delta x}{(x+\delta x)x}. \end{aligned}$$

Then, dividing by δx gives

$$\frac{\delta y}{\delta x} = \frac{-1}{(x+\delta x)x},$$

from which

$$\lim_{\delta x \to 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx} = -\frac{1}{x^2}.$$

Composite Functions

3. Find dy/dx when $y = (x^2 - 5x + 7)^4$.

Answer. Define $u = (x^2 - 5x + 7)$. Then $y = u^4$, and hence

$$\frac{dy}{du} = 4u^3 \qquad \text{and} \qquad \frac{du}{dx} = 2x - 5.$$

By using the chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4(x^2 - 5x + 7)^3(2x - 5).$$

4. Find dy/dx when $y = (\sqrt{x} - 1/\sqrt{x})^5$.

Answer. Define $u = (\sqrt{x} - 1/\sqrt{x}) = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. Then $y = u^5$, whence

$$\frac{dy}{du} = 5u^4 \qquad \text{and} \qquad \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}.$$

Hence, using the chain rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{5}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^4 \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x^3}}\right).$$

5. Find the derivative of the function $f(x) = (2 - x^4)^{-3}$.

Answer. Define $u = 2 - x^4$ and $y = u^{-3}$. Then

$$\frac{dy}{du} = -3u^{-4} \qquad \text{and} \qquad \frac{du}{dx} = -4x^3.$$

Hence, using the chain rule, we find that the derivative of the function is

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{12x^3}{(2 - x^4)^4}.$$

6. Differentiate $\sqrt{(1 + x^{-1})}$.

Answer. Define $y = u^{\frac{1}{2}}$ with $u = 1 + x^{-1}$. Then

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \qquad \text{and} \qquad \frac{du}{dx} = -x^{-2}.$$

Then the derivative of the function is found via the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{2x^2\sqrt{1 + x^{-1}}}.$$

Products and Quotients

7. Differentiate $y = (2x + 1)^3(x - 8)^7$ with respect to x.

Answer. Define $u = (2x + 1)^3$ and $v = (x - 8)^7$. Then

$$\frac{du}{dx} = 6(2x + 1)^2 \qquad \text{and} \qquad \frac{dv}{dx} = 7(x - 8)^6;$$

whence the derivative of $y = uv$ is found via the product rule:

$$\begin{aligned} \frac{d}{dx}(uv) &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 7(2x + 1)^3(x - 8)^6 + 6(x - 8)^7(2x + 1)^2 \\ &= 5(2x + 1)^2(x - 8)^6(4x - 11). \end{aligned}$$

8. Find the derivative of $\sqrt{(x + 3)^3(x - 1)^4}$.

Answer. Define $y = \sqrt{p} = \sqrt{uv}$, with $u = (x + 3)^3$ and $v = (x - 1)^4$. Then

$$\frac{dy}{dp} = \frac{1}{2\sqrt{p}}, \qquad \frac{du}{dx} = 3(x + 3)^2 \qquad \text{and} \qquad \frac{dv}{dx} = 4(x - 1)^3,$$

whence the product rule gives

$$\begin{aligned} \frac{dp}{dx} &= \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \\ &= 4(x + 3)^3(x - 1)^3 + 3(x - 1)^4(x + 3)^2 \\ &= (x + 3)^2(x - 1)^3(7x + 9). \end{aligned}$$

Applying the chain rule gives the derivative in question:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dp} \times \frac{dp}{dx} \\ &= \frac{1}{2}\frac{(x + 3)^2(x - 1)^3(7x + 9)}{\sqrt{(x + 3)^3(x - 1)^4}} \\ &= \frac{1}{2}(x - 1)(7x + 9)\sqrt{(x + 3)}. \end{aligned}$$

9. Differentiate $3x^2/(x-1)^4$ with respect to x.

Answer. Let $y = u/v$ with $u = 3x^2$ and $v = (x-1)^4$. Then

$$\frac{du}{dx} = 6x \qquad \text{and} \qquad \frac{dv}{dx} = 4(x-1)^3,$$

and so the quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{v(du/dx) - u(dv/dx)}{v^2} \\ &= \frac{6x(x-1)^4 - 12x^2(x-1)^3}{(x-1)^8} \\ &= -\frac{6x(x-1)^3(x+1)}{(x-1)^8} = -\frac{6x(x+1)}{(x-1)^5}.\end{aligned}$$

10. Differentiate $\sqrt{(x-3)/(x^2+2)}$ with respect to x.

Answer. Define $y = u/v$ with $u = (x-3)^{\frac{1}{2}}$ and $v = (x^2+2)^{\frac{1}{2}}$. Then

$$\frac{du}{dx} = \frac{1}{2}(x-3)^{-\frac{1}{2}} \qquad \text{and} \qquad \frac{dv}{dx} = x(x^2+2)^{-\frac{1}{2}},$$

and so the quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{v(du/dx) - u(dv/dx)}{v^2} \\ &= \frac{\frac{1}{2}(x^2+2)^{\frac{1}{2}}(x-3)^{-\frac{1}{2}} - x(x-3)^{\frac{1}{2}}(x^2+2)^{-\frac{1}{2}}}{x^2+2}.\end{aligned}$$

Next, multiplying top and bottom of this expression by $2(x-3)^{\frac{1}{2}}(x^2+2)^{\frac{1}{2}}$ and simplifying gives

$$\frac{dy}{dx} = \frac{6x - x^2 + 2}{(x^2+2)\sqrt{(x-3)(x^2+2)}}.$$