INTRODUCTORY ECONOMETRICS : Exercise 3 (Assessed)

Matrix Algebra and National Income Accounts

The following tableau represents the national income accounts of a closed economy of which the productive activities are assigned to three sectors:

	Agriculture	*Industry*	*Services*	*Final Demand*	*Total Demand*
Agriculture	X_{AA}	X_{AM}	X_{AS}	Y_A	X_A
Industry	X_{MA}	X_{MM}	X_{MS}	Y_M	X_M
Services	X_{SA}	X_{SM}	X_{SS}	Y_S	X_S
Factors	V_A	V_M	V_S	$V = Y$	
Activity Level	X_A	X_M	X_S		

The following elements are defined:

X_{ij} is the value of the goods shipped from the ith sector to the jth sector,

Y_i is the final demand, of consumers and investors, for the output of the ith sector,

V_j is the value added to the output of the jth sector by the services of the factors of production which correspond to the earnings of Workers, the income of Rentiers and Capitalists, and the profits of Entrepreneurs.

In addition, we have the following accounting identities:

$X_i = \sum_j X_{ij} + Y_i$: the total output of the ith sector consists of the output X_{ii} which is used within the sector, the outputs $X_{ij}; j \neq i$ which are shipped to other sectors, and the output Y_i which satisfies the final demand,

$X_j = \sum_i X_{ij} + V_j$: the input to the jth sector is equal to the sum of the inputs from all sectors—including the goods shipped from one firm within the sector to another—plus the factor inputs V_j,

$V = \sum_j V_j = Y = \sum_i Y_i$: there are two ways of reckoning the final product of the economy. We may sum the factors inputs or we may sum the final demands. The two sums should be equal.

The input–output analysis of Leontief depends upon the following fundamental postulate:

> The value of the goods shipped from the ith sector to the jth sector is directly proportional to the level of activity in the jth sector. That is to say, we have $X_{ij} = a_{ij}X_j$ for some constant input–output coefficient a_{ij}. Likewise, the value of the factor inputs to the jth sector is $V_j = f_jX_j$, where f_j is a constant.

It follows from this postulate that we can write the first of the accounting identities in the form of

$$X_i = \sum_j a_{ij}X_j + Y_i.$$

Now let us define the vectors $x = [X_A, X_M, X_S]'$, $y = [Y_A, Y_M, Y_S]'$ and the matrix $A = [a_{ij}]$. Then, if we apply the first of the accounting identities to all three sectors, we derive an equation in the form of

$$\begin{bmatrix} X_A \\ X_M \\ X_S \end{bmatrix} = \begin{bmatrix} a_{AA} & a_{AM} & a_{AS} \\ a_{MA} & a_{MM} & a_{MS} \\ a_{SA} & a_{SM} & a_{SS} \end{bmatrix} \begin{bmatrix} X_A \\ X_M \\ X_S \end{bmatrix} + \begin{bmatrix} Y_A \\ Y_M \\ Y_S \end{bmatrix}.$$

In summary notation, this becomes

$$x = Ax + y, \qquad \text{or, equivalently,} \qquad (I - A)x = y.$$

The solution, in terms of the vector of activity levels, is provided by

$$\begin{aligned} x &= (I - A)^{-1} y \\ &= (I + A + A^2 + A^3 + \cdots)y. \end{aligned}$$

The series form $(I - A)^{-1} = (I + A + A^2 + A^3 + \cdots)$ for the inverse of $I - A$ is available only on account of the specialised nature of the matrix A.

The series form of the solution has an intuitive interpretation. The first term y stands for the output which satisfies the final demand. The second term Ay stands for the immediate input requirements of the sectors which endeavour to satisfy the final demand. The third term A^2y stands for the output which is required in the production of the intermediate output Ay. The story continues ad infinitum—but the elements of the generic term $A^n y$ become negligible as $n \to \infty$.

Let us take the following numerical version of the tableau:

	Agriculture	*Industry*	*Services*	*Final Demand*	*Total Demand*
Agriculture	50	50	50	100	250
Industry	80	100	20	180	380
Services	50	60	30	110	250
Factors	70	170	150	390	
Activity Level	250	380	250		

For an exercise, you are invited to perform the following tasks using *Excel 4.0*:

(1) Confirm that the various accounting identities are satisfied by the elements of the table.

(2) Derive the 3×3 matrix $A = [a_{ij}]$ of input–output coefficients.

(3) Calculate the partial sums $S_2 = I+A$, $S_3 = I+A+A^2$ and $S_4 = I+A+A^2+A^3$ in order to determine the rate of convergence of the series $\{I + A + A^2 + \cdots\} = (I - A)^{-1}$.

(4) Determine the activity levels in each of the sectors and find the values of the necessary factor inputs when the final demand is given by $[Y_A, Y_M, Y_S] = [120, 200, 150]$.