# MULTICHANNEL FILTER BANKS

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This lecture describes how the two-channel filter bank may be generalised to create an $M$-channel filter bank in which the frequency bands of the channels are of equal widths.

The battery of $M$ filters will span the Nyquist frequency range of $[-\pi, \pi]$. The output of each filter is critically downsampled by a factor of $M$ to create a minimal representation of its information contents.

Before embarking on the analysis, we must consider the means of generating the $M$ filters from a single prototype filter. It is also necessary to demonstrate the effects, in the frequency domain, of downsampling by a factor of $M$.

After being subjected to downsampling, the outputs of the filters may be upsampled and smoothed via a second set of filters, described as the synthesis filters. Thereafter, the separate signals can be recombined to reconstruct the input signal.

To achieve a perfect reconstruction, it is necessary to eliminate the aliasing effects, which correspond to the mutual interference of channels.

## Frequency Shifting

If the Fourier transform of $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ is $\xi(\omega) = \sum x_t e^{-\mathrm{i}\omega t}$, then the Fourier transform of $x(t).\exp\{\mathrm{i}\gamma t\} = \{x_t e^{i\gamma t}\}$ is $\xi(\omega - \gamma) = \sum x_t e^{-\mathrm{i}(\omega-\gamma)t}$; and the modulation shifts the centre of the frequency-domain function from $\omega = 0$ to $\omega = \gamma$.

Given that $z = e^{-\mathrm{i}\omega t}$, the $z$-transform of the original sequence can be written as $x(z) = \sum_t x_t z^t$ and that of the modulated sequence as $x(ze^{\mathrm{i}\gamma}) = \sum_t x_t (ze^{\mathrm{i}\gamma})^t$

Given that $\cos(\gamma t) = (e^{\mathrm{i}\gamma t} + e^{-\mathrm{i}\gamma t})/2$, it follows that the Fourier transform of $\cos(\gamma t).x(t)$ is $\{\xi(\omega - \gamma) + \xi(\omega + \gamma)\}/2$, and the result can be represented by writing

$$2\cos(\gamma t).x(t) \longleftrightarrow \xi(\omega - \gamma) + \xi(\omega + \gamma).$$

To create a battery of $M$ filters, equally-spaced in the frequency range $[-\pi, \pi]$, a lowpass prototype filter $H(z)$ on $[-\pi/2M, \pi/2M]$ is translated to the following centres:

$$\pm\gamma_j = \pm\pi(j + 1/2)/M; j = 0, 1, \ldots, M - 1.$$

Then, the $j$th filter is

$$\begin{aligned} H_j(z) &= H(ze^{\mathrm{i}\gamma_j}) + H(ze^{-\mathrm{i}\gamma_j}) \\ &= H_j^+(z) + H_j^-(z). \end{aligned}$$

## Downsampling

Consider the $z$-transform of the sequence $y(t)$:

$$y(z) = \sum_t y_t z^t = \cdots + y_0 + y_1 z + y_2 z^2 + y_3 z^3 + \cdots.$$

Downsampling by a factor of 3 gives:

$$y_0(z) = \sum_t y_{3t} z^t \quad = \cdots + y_0 + y_3 z + y_6 z^2 + y_9 z^3 + \cdots$$

The downsampled subseries is expressed in terms of the original series $y(z)$ using the delta function $\delta(t \bmod 3)$, which eliminates the terms for which the index $t$ is not a multiple of 3. Thus,

$$y_j(z) = \sum_t y_{t+j} \delta(t \bmod 3) z^{t/3}; \quad j = 0, 1, 2.$$

The delta function can be expressed as a sum of complex exponentials. In the case of $M = 3$, the complex exponential is $W_3 = \exp\{\mathrm{i}2\pi/3\}$. Then

$$\delta(t \bmod 3) = \frac{1}{3} \sum_{k=0}^{2} W_3^{kt} = \frac{1}{3} \sum_{k=0}^{2} e^{\mathrm{i}2\pi kt/3}.$$

## The Delta Function

The delta function has the property that

$$\delta(t \bmod M) = \begin{cases} 1, & \text{if } t \bmod M = 0, \\ 0, & \text{otherwise.} \end{cases}$$

This is achieved by specifying that

$$\delta(t \bmod M) = \frac{1}{M}\sum_{j=0}^{M-1} W_M^{tj},$$

wherein $W_M = \exp\{-\mathrm{i}2\pi/M\}$ is a periodic function such that $W_M^q = W_M^{q \bmod M}$.

The essential results are that

$$\sum_{j=0}^{M-1} W_M^{tj} = \frac{1 - W_M^{Mt}}{1 - W_M^t} = \begin{cases} 0, & \text{if } t \neq 0, \\ M, & \text{if } t = 0. \end{cases}$$

The first equality follows from the fact that, in the numerator of the RHS, there is $W_M^{Mt} = W^0 = 1$, when $t \neq 0$.

The second follows from the fact that, on the LHS, there is $W_M^0 = 1$, when $t = 0$.

## The Channels of the Filter Bank

Let $H_j(z); j = 0, \ldots, M-1$ be the analysis filters and let $F_j(z); j = 0, \ldots, M-1$ be the synthesis filters. In the absence of the upsampling and downsampling, the recombined signal would be

$$x(z) = \sum_{j=0}^{M-1} F_j(z)H_j(z)y(z).$$

With downsampling by a factor of $M$, the outputs of the analysis sections of the filter bank become

$$\frac{1}{M}\sum_{k=0}^{M-1} H_j(z^{1/M}W_M^k)y(z^{1/M}W_M^k); \quad j = 1, \ldots, M-1,$$

wherein $W_M = \exp\{-\mathrm{i}2\pi/M\}$ and then, with upsampling by a factor of $M$ and smoothing, they become

$$\frac{1}{M}F_j(z)\sum_{k=0}^{M-1} H_j(zW_M^k)y(zW_M^k); \quad j = 1, \ldots, M-1.$$

Hereafter, for notational convenience, we may omit the subscript from $W_M$.

## The Reconstruction of the Signal

In the case of $M = 3$, the reconstructed signal is

$$x(z) = \frac{1}{3}\left[\,F_0(z) \quad F_1(z) \quad F_2(z)\,\right] \times \begin{bmatrix} H_0(z) & H_0(zW) & H_0(zW^2) \\ H_1(z) & H_1(zW) & H_1(zW^2) \\ H_2(z) & H_2(zW) & H_2(zW^2) \end{bmatrix} \begin{bmatrix} y(z) \\ y(zW) \\ y(zW^2) \end{bmatrix}.$$

When the matrix multiplications are performed, this gives

$$x(z) = \frac{1}{3}\left(\sum_{j=0}^{2} F_j(z)H_j(z)\right) y(z) + \frac{1}{3}\sum_{j=0}^{2}\sum_{k=1}^{2} F_j(z)H_j(zW^k)y(zW^k).$$

The second collection of the terms is attributable to the effects of aliasing and imaging that are the results of downsampling.

When $M = 3$, there are two alias cancellation conditions:

$$F_0(z)H_0(zW) + F_1(z)H_1(zW) + F_2(z)H_2(zW) = 0,$$

$$F_0(z)H_0(zW^2) + F_1(z)H_1(zW^2) + F_2(z)H_2(zW^2) = 0.$$

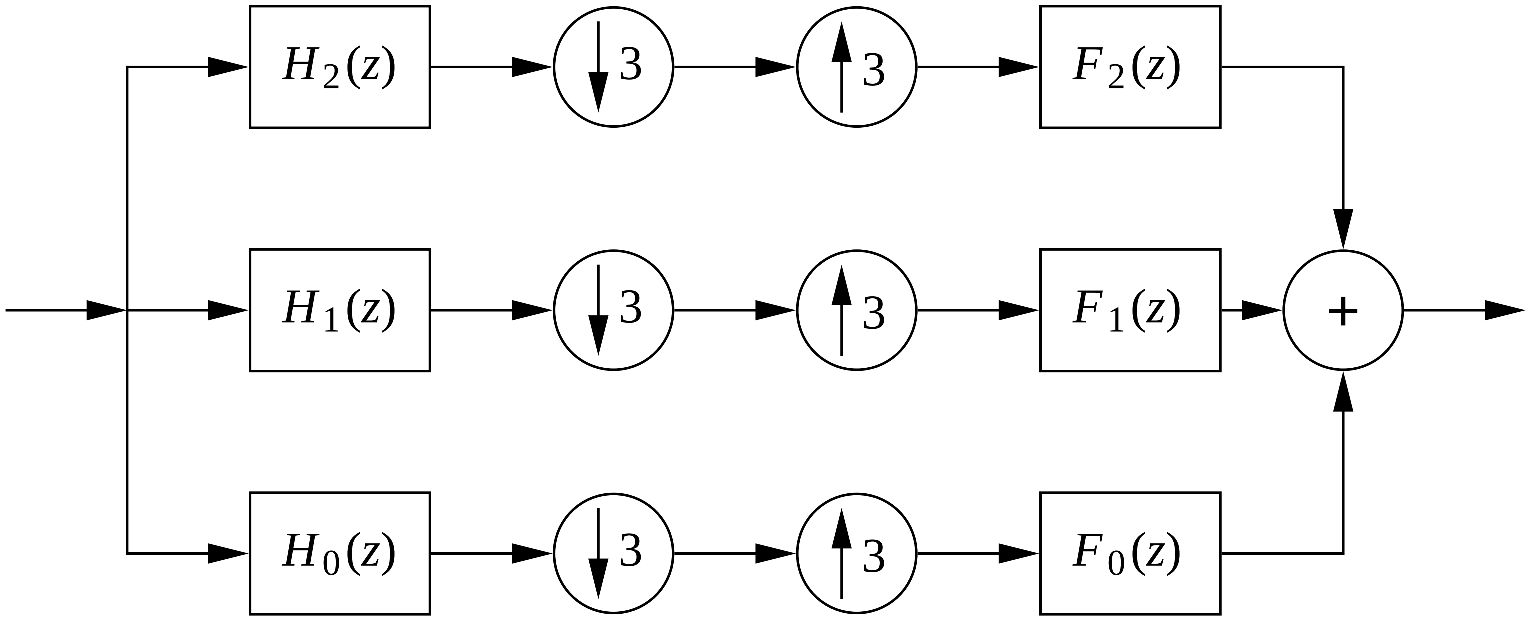


**Figure 1.** A depiction of the three-channel filter bank.

## Aliasing Cancelations: the Case of M=3

Figure 2 illustrates the case of $M = 3$. Given that the transition bands of the filters occur within intervals not exceeding $\pi/M$, aliasing cancellation can be achieved by the interaction of adjacent filters.

In Figure 2, the aliasing effects in the first tranche, which are in the vicinity of $\pm\pi/3$ are cancelled by the effects at the same frequencies in the second tranche.

The effects at $\pm 2\pi/3$ in the second tranche are cancelled by the effects at these frequencies within the third tranche.

It is to be presumed that, apart from possible changes of their signs, the synthesis filters are equal to the analysis filters.

The filters depicted in the diagram are power complementary, which is necessary for perfect reconstruction:

$$\frac{1}{3}\sum_{j=0}^{2} H_j(z)H_j(z^{-1}) = 1.$$

Dilating these filters by factor of 3 will give rise to uniform spectra over the interval $[-\pi, \pi]$, which is a sufficient condition for their sequential orthogonality.

The lateral orthogonality of the filters is not guarantee.

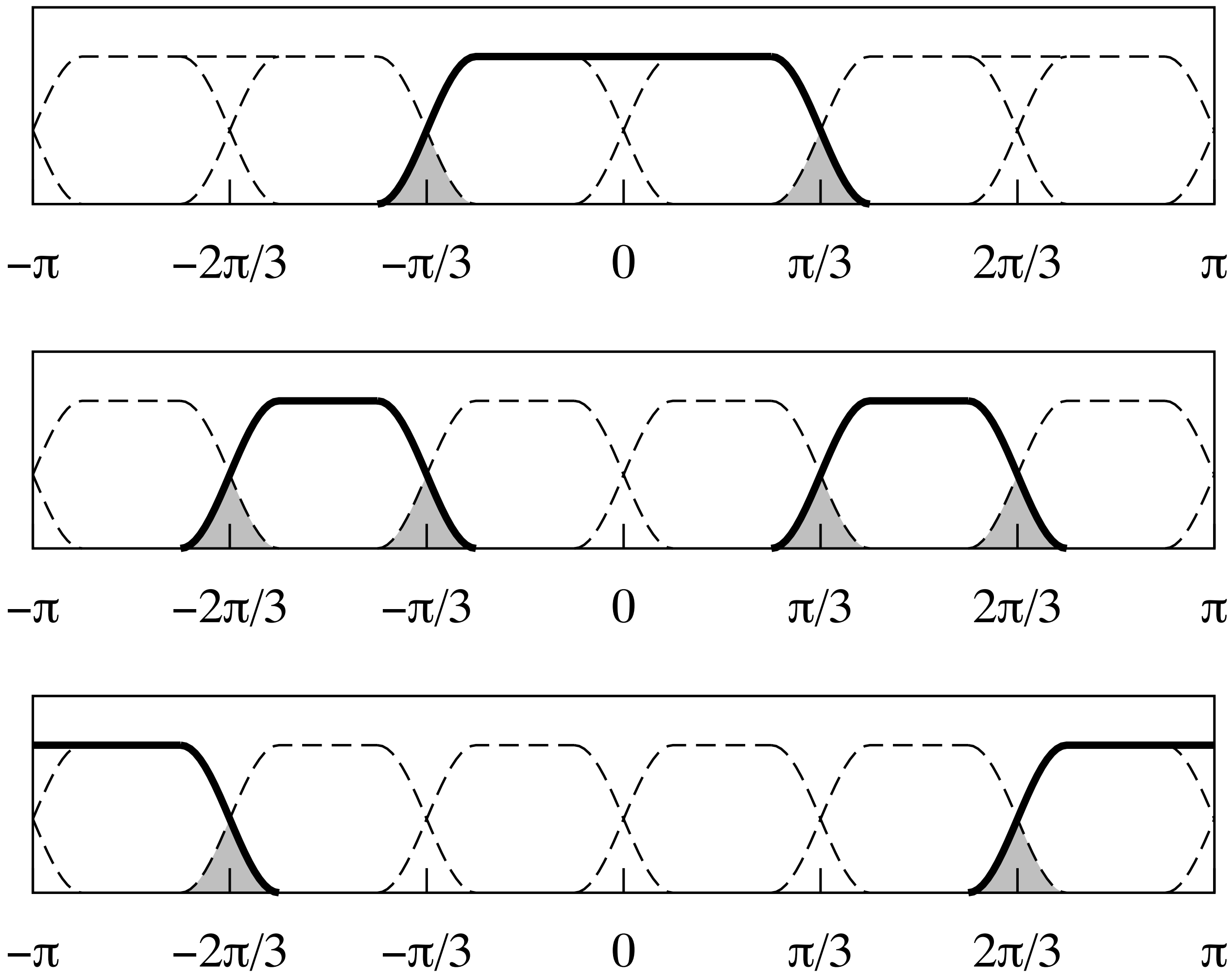


**Figure 2.** Aliasing cancellation is achieved by the interaction of adjacent filters.

## Multiplexing in Time and Frequency

Time-division multiplexing, which was first used extensively at the beginning of the 20th century, allows multiple signals to be transmitted simultaneously via a single channel by upsampling and interleaving the signals.

In frequency-division multiplexing, multiple signals are transmitted through a single medium by assigning them to the different carrier frequencies. Analogue radio broadcasting has used frequency-division multiplexing to separate the transmissions of numerous radio stations.

Transmultiplexing combines time-division and frequency-division multiplexing. Figure 3 shows the structure of a transmultiplexer.

The input signals are upsampled and filtered by a synthesis bank of filters and then combined. The combined signal is filtered by a battery of analysis filters and downsampled to produce separate signals that are intended to replicate the multiple input signals.

The transmultiplexer is the dual of the $M$-channel filter bank.

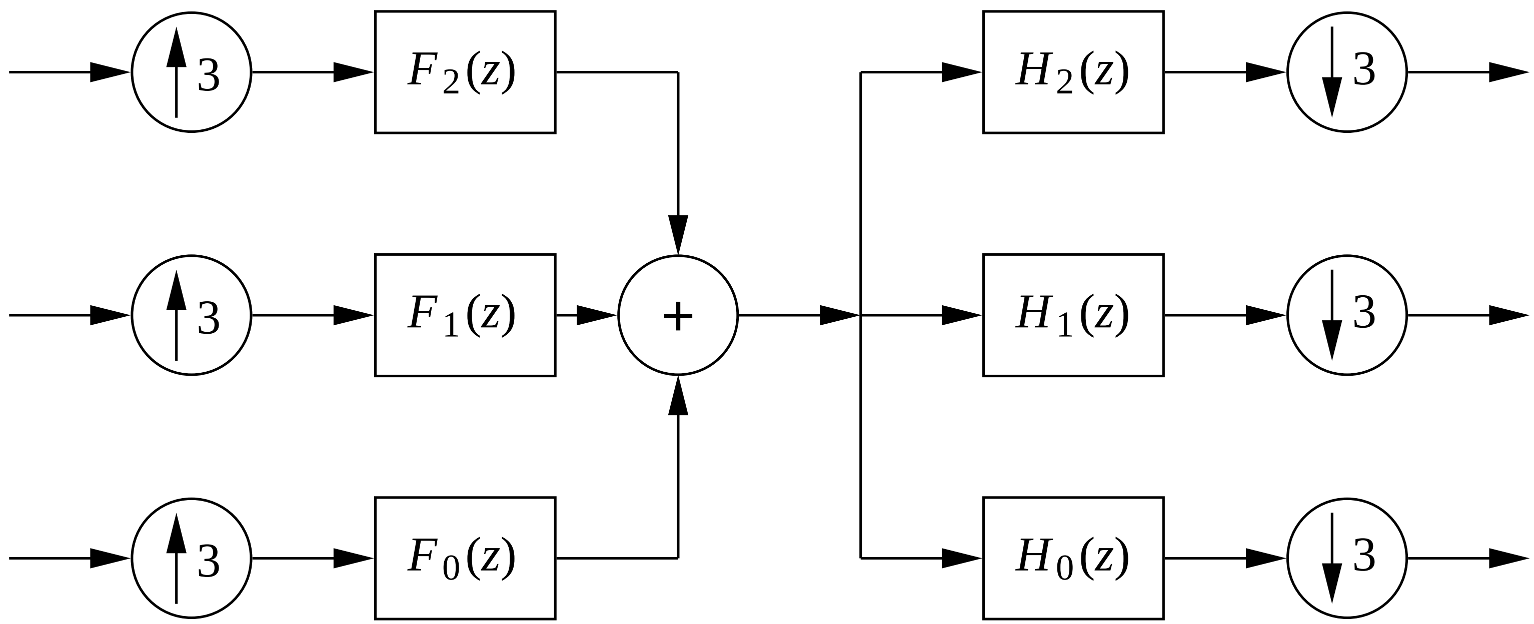


**Figure 3.** A three-channel transmultiplexer.

## The Polyphase Formulation

Much of the theory of $M$-channel filter banks has been developed within the context of the polyphase formulation. In the case of $M = 3$, the three phases of the data are

$$\begin{aligned} y_0(z) &= \sum_t y_{3t} z^t &&= \{\cdots + y_0 + y_3 z + y_6 z^2 + y_9 z^3 + \cdots\}, \\ y_1(z) &= \sum_t y_{3t+1} z^t &&= \{\cdots + y_1 + y_4 z + y_7 z^2 + y_{10} z^3 + \cdots\}, \\ y_2(z) &= \sum_t y_{3t+2} z^t &&= \{\cdots + y_2 + y_5 z + y_8 z^2 + y_{11} z^3 + \cdots\}, \end{aligned}$$

$$y(z) = y_0(z) + z y_1(z) + z^2 y_2(z).$$

The polyphase expansion of the analysis filters is

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} E_{0,0}(z^3) & E_{0,1}(z^3) & E_{0,2}(z^3) \\ E_{1,0}(z^3) & E_{1,1}(z^3) & E_{1,2}(z^3) \\ E_{2,0}(z^3) & E_{2,1}(z^3) & E_{2,2}(z^3) \end{bmatrix} \begin{bmatrix} 1 \\ z \\ z^2 \end{bmatrix}.$$

The synthesis filters can also be expanded via a polyphase decomposition:

$$\begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} = \begin{bmatrix} R_{0,0}(z^3) & R_{1,0}(z^3) & R_{2,0}(z^3) \\ R_{0,1}(z^3) & R_{1,1}(z^3) & R_{2,1}(z^3) \\ R_{0,2}(z^3) & R_{1,2}(z^3) & R_{2,2}(z^3) \end{bmatrix} \begin{bmatrix} z^2 \\ z \\ 1 \end{bmatrix}.$$

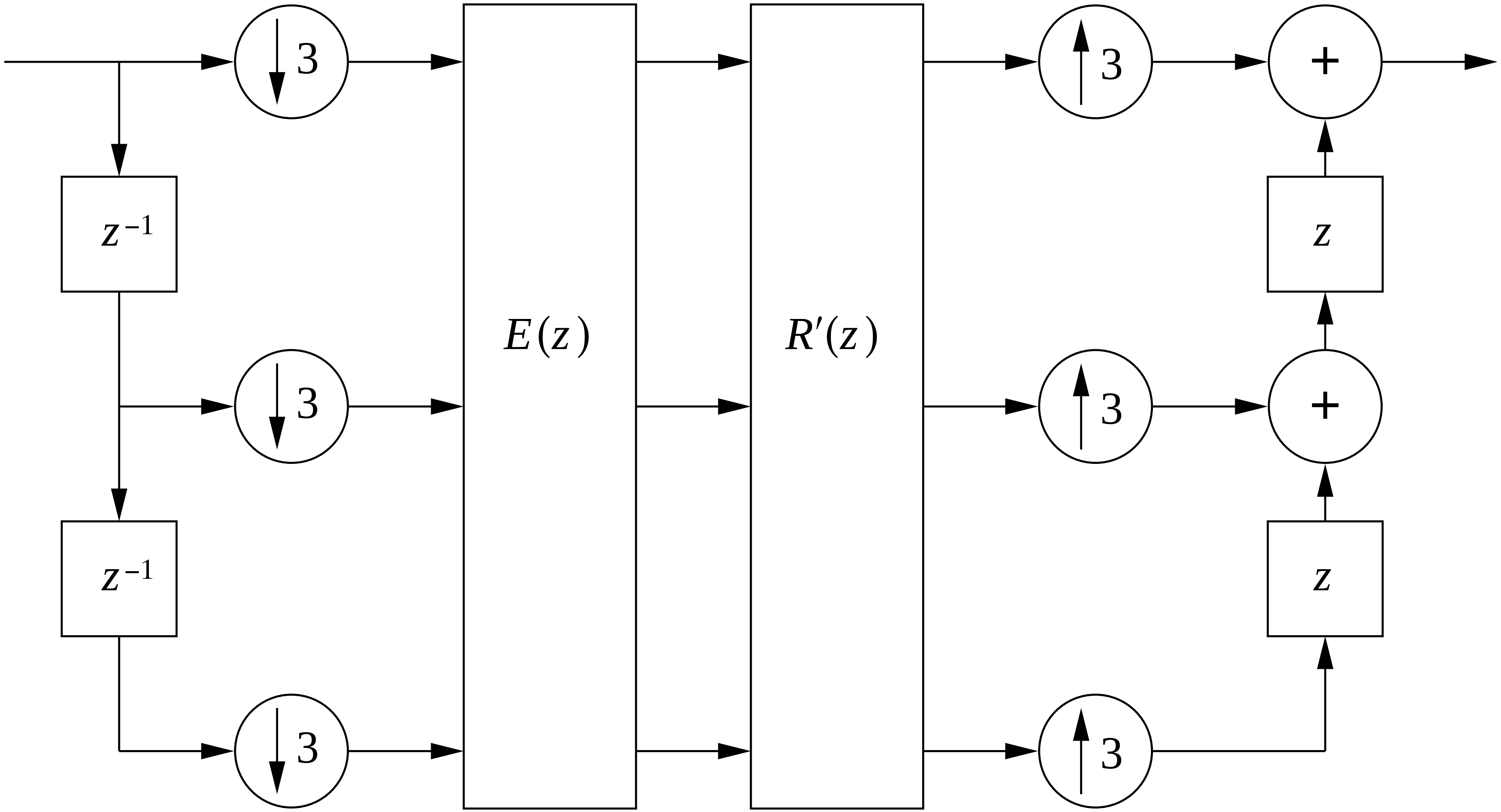


**Figure 4.** A three-channel filter bank can be constructed that separates the data into three phases.

## The Block-Orthogonal Transform

An alternative approach to multichannel filter banks originates in the block-orthogonal transforms that are associated with monochrome digital pictures, represented by a matrix $P = [p_{ij}]$

If $P = \alpha\beta'$ is the product of a single row and column, then the transformed image is $C = TPT' = \gamma\delta$, with $\gamma = T\alpha$ and $\delta = T\beta$.

The object of the transform would be to reduce the correlation between picture elements, enabling a compression by setting some of them to zeros. If $T$ is an orthonormal matrix, such that $T'T = TT' = I$, then $P = T'CT$ can be readily recovered from $C$.

A more complicated image $P = [P_{ij}; i, j = 0, \ldots, N-1]$, consisting of $N^2$ rectangular sub-images, can be transformed, one subimage at a time, into $C = [C_{ij}] = [Q'P_{ij}Q]$ via

$$T = \begin{bmatrix} Q' & 0 & 0 & \ldots & 0 & 0 \\ 0 & Q' & 0 & \ldots & 0 & 0 \\ 0 & 0 & Q' & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & Q' & 0 \\ 0 & 0 & 0 & \ldots & 0 & Q' \end{bmatrix}.$$

## Lapped Orthogonal Transforms

A disadvantage of a block-orthogonal transform are the disjunctions that are liable to occur between adjacent subimages.

It has been proposed that the transformation matrix $T$ should be formed from overlapping oblong blocks, allowing a smoother transition between subimages.

A circulant version of a lapped transformation is

$$T = \begin{bmatrix} Q_1' & Q_0' & 0 & \ldots & 0 & 0 \\ 0 & Q_1' & Q_0' & \ldots & 0 & 0 \\ 0 & 0 & Q_1' & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & Q_1' & Q_0' \\ Q_0' & 0 & 0 & \ldots & 0 & Q_1' \end{bmatrix}.$$

The matrix $[Q_1', Q_0']$ may be described as a block transformation.

The conditions $T'T = TT' = I$, which are necessary for the recovery of the data, imply that

$$Q_0'Q_0 + Q_1'Q_1 = Q_0Q_0' + Q_1Q_1' = I,$$

and that

$$Q_0'Q_1 = Q_1'Q_0 = 0 \quad \text{and} \quad Q_0Q_1' = Q_1Q_0' = 0.$$