DYADIC WAVELETS ANALYSIS

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The previous lecture has described the dyadic decomposition of sampled data systems. Now, we shall consider the continuous-time processes that underlie the data.

Since no assumptions will be made regarding the statistical homogeneity of the processes, a global parametric description of its properties is not available. Instead, a set of basis functions is required that can capture the local variations of the data.

The basis functions are provided by sequences of continuous wavelets and scaling functions that are ordered both in time and in frequency.

The coefficients associated to the basis functions are the sequences obtained by the procedures of filtering and dowsampling previously described.

A matrix representation of the dyadic decomposition will be developed to replace the z-transform representations. In dealing with finite data sequences, we shall resort to circulant matrices.

The Dyadic Decomposition of a Space of Functions

It is assumed that the data sequence $\{y_k; k = 0, 1, 2, \ldots, T-1\}$ has been sampled from a continuous, or piecewise continuous, function $f(t)$, with $t \in [0, T)$.

The function can be reconstituted, approximately, by associating a scaling function $\phi_{0,k}(t) = \phi(t-k)$ to each of the data points and by summing the result:

$$f(t) \simeq y(t) = \sum_{k=0}^{T-1} y_k \phi(t-k).$$

The scaling function $\phi(t-k)$ is a localised function defined on the real line $\mathcal{R}$. Its nominal centre is at $t = k$; and successive values of k represent successive displacements to the right.

The scaling functions constitute an orthonormal basis of the space $\mathcal{V}_0$ in which the function $f(t)$, or its approximation $y(t)$, resides. Therefore,

$$\int_t \phi(t-j)\phi(t-k)dt = \langle \phi(t-j), \phi(t-k) \rangle = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

The individual elements of the data sequence, which are the amplitude coefficients of the associated scaling functions, are given back by

$$y_k = \int_t y(t)\phi(t-k)dt = \langle y(t), \phi(t-k) \rangle.$$

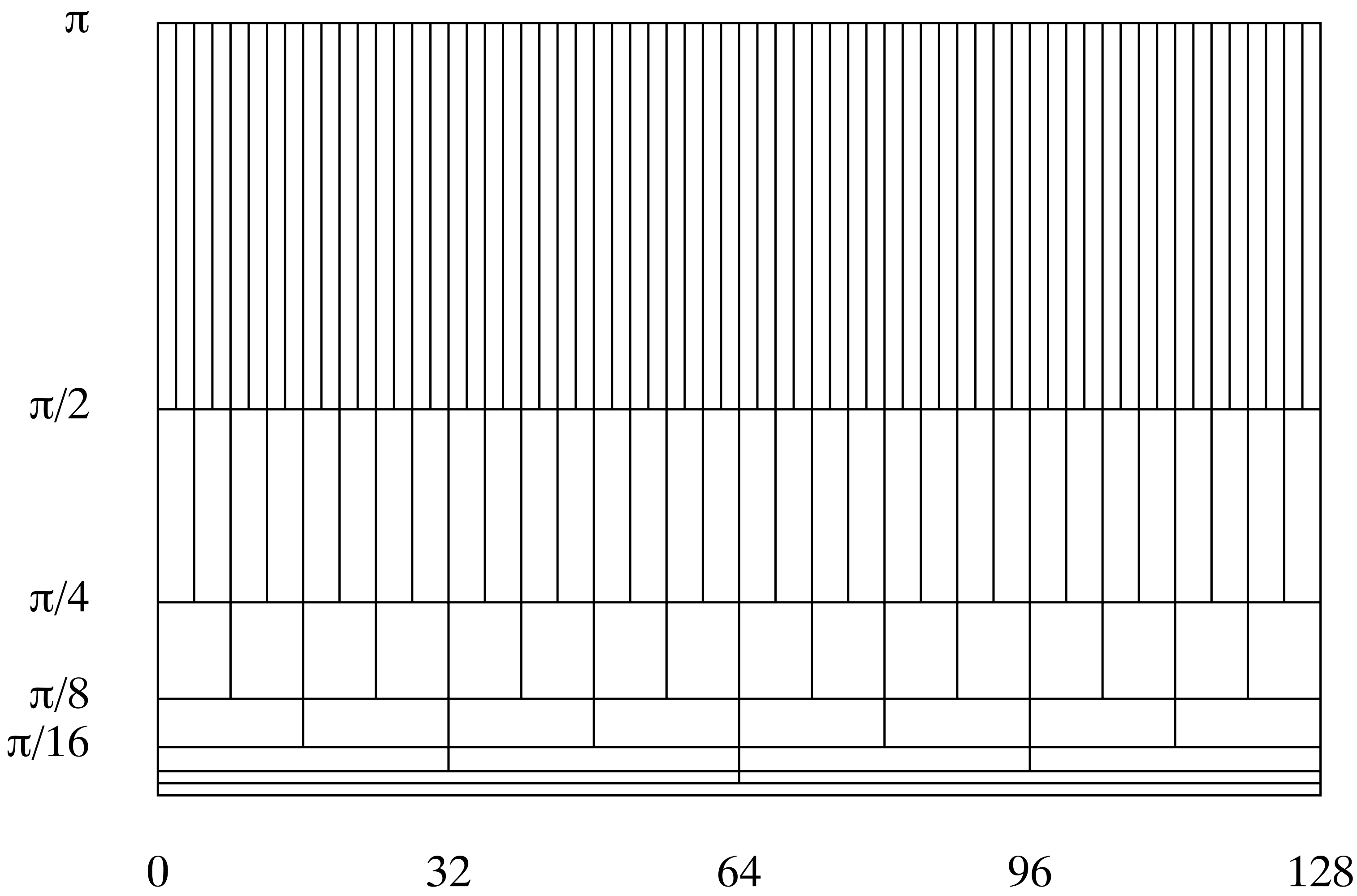

Figure 1. The partitioning of the time–frequency plane according to a multiresolution analysis of a data sequence of $128 = 2^7$ points.

The Initial Basis and the Derived Basis

The scaling functions $\phi_{0,k}(t) = \phi(t-k); k = 0, 1, \ldots, T-1$, of the initial basis have nominal frequency contents that extend from a limiting frequency of π radians per sampling interval down to the zero frequency.

In a dyadic wavelets analysis, the T amplitude coefficients of the initial basis, which are the sampled values, are transformed into a hierarchy of T coefficients associated with an alternative basis, that is ordered both according to the temporal locations of the wavelets and according to their frequency contents.

The hierarchy of wavelets within the final basis can be described with reference to a so-called mosaic diagram that partitions the time-frequency plane, which corresponds to the space $\mathcal{V}_0$.

In the diagram, the height of a cell corresponds to a bandwidth in the frequency domain, whereas its width denotes a temporal duration.

Centred on each cell, but liable to extend beyond its temporal boundaries, there is a wavelet. The frequency contents of the wavelet is also liable to extend beyond the nominal bandwidth that is indicated in the figure.

The Dyadic Decomposition of a Space of Functions

The horizontal bands of the mosaic diagram are obtained by successive divisions of the frequency range $[0, \pi]$ and, therefore, of the function space $\mathcal{V}_0$:

$$
\begin{array}{rcl}
[0, \pi] & \longrightarrow & [0, \pi/2], [\pi/2, \pi] \\
[0, \pi/2] & \longrightarrow & [0, \pi/4], [\pi/4, \pi/2] \\
\vdots & & \vdots \qquad\qquad \vdots \\
[0, \pi/2^{n-1}] & \longrightarrow & [0, \pi/2^n], [\pi/2^n, \pi/2^{n-1}]
\end{array}
\qquad
\begin{array}{rcl}
\mathcal{V}_0 & \longrightarrow & \mathcal{V}_1 \oplus \mathcal{W}_1, \\
\mathcal{V}_1 & \longrightarrow & \mathcal{V}_2 \oplus \mathcal{W}_2. \\
\vdots & & \vdots \\
\mathcal{V}_{n-1} & \longrightarrow & \mathcal{V}_n \oplus \mathcal{W}_n.
\end{array}
$$

The subspaces are mutually orthogonal with $\mathcal{W}_j \perp \mathcal{V}_j$ and $\mathcal{W}_j \oplus \mathcal{V}_j = \mathcal{V}_{j-1}$. Thus, any element in $y(t) \in \mathcal{V}_0$ can be expressed as $y(t) = w_1(t) + v_1(t)$ with $w_1(t) \in \mathcal{W}_1$ and $v_1(t) \in \mathcal{V}_1$.

If $\mathcal{V}_{j-1}$ contains $T/2^{j-1}$ basis functions, then $\mathcal{V}_j$ and $\mathcal{W}_j$ will both contain $T/2^j$ basis functions, which is half as many.

The ultimate decomposition is

$$\mathcal{V}_0 = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots \oplus \mathcal{W}_n \oplus \mathcal{V}_n.$$

The Decomposition of Continuous Function and of a Sequence

Given the decomposition of $\mathcal{V}_0$ as a sum of mutually orthogonal subspaces, it is possible to represent $y(t) \in \mathcal{V}_0$ as a sum of orthogonal components. Thus

$$y(t) = w_1(t) + w_2(t) + \cdots + w_n(t) + v_n(t),$$

with $w_j(t) \in \mathcal{W}_j$ for $j = 1, \ldots, n$ and with $v_n(t) \in \mathcal{V}_n$.

The component $w_j(t)$ is represented, in terms of the basis functions of $\mathcal{W}_j$, by

$$w_j(t) = \sum_{k=0}^{[T/2^j]-1} \beta_{jk}\psi_{j,k}(t),$$

where, β_{jk} is the amplitude coefficient of the kth wavelet function.

The sampled ordinates of $y(t) \in [0, T]$ are

$$y_k = w_{1k} + w_{2k} + \cdots + w_{nk} + v_{nk}; \quad k = 0, 1, \ldots, T-1,$$

whence

$$y(t) = \sum_k y_k\phi_{0,k}(t) = \sum_{k=0}^{T-1} \Big\{ \sum_{j=1}^{n} w_{jk} + v_{nk} \Big\} \phi_{0,k}(t).$$

Successive Dilations of the Basis Functions

The scaling functions and the wavelets in successive bands are versions of the functions in the bands above, dilated by a factor of 2. The basis functions of $\mathcal{V}_0$ and of $\mathcal{V}_1 \subset \mathcal{V}_0$ are, repectively,

$$\phi_{0,k}(t) = \phi(t-k) \in \mathcal{V}_0 \qquad \text{and} \qquad \phi_{1,k}(t) = 2^{-1/2}\phi(2^{-1}t-k) \in \mathcal{V}_1.$$

More generally,

$$\phi_{j,k}(t) = 2^{-j/2}\phi(2^{-j}t-k).$$

Multiplying the functions by the factor $2^{-j/2}$ ensures that their squares will continue to integrate to unity.

Also observe that the basis functions of $\mathcal{V}_1$ are separated one from the next by intervals of 2 points.

Thus, whereas $\phi_{1,k}(t) = 2^{-1/2}\phi(2^{-1}t-k)$ will have its centre at the point $t = 2k$, which is the solution of $2^{-1}t - k = 0$, the succeeding function $\phi_{1,k+1}(t)$ will have its centre at the point $t = 2k + 2$.

Successive wavelet functions bear an analogous relationship; and there is

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t-k).$$

The Two-Scale Dilation Equations

It is possible to express the scaling function $\phi_{1,0}(t) \in \mathcal{V}_1$ as a linear combination of the elements of the basis of a space $\mathcal{V}_0$ of twice the resolution. Thus

$$\phi_{1,0}(t) = 2^{-1/2}\phi(2^{-1}t) = \sum_k g_k \phi(t-k),$$

were

$$g_k = \langle \phi_{1,0}(t), \phi(t-k) \rangle = \int_{-\infty}^{\infty} \phi_{1,0}(t)\phi(t-k)dt.$$

More generally, the relationship between the basis elements of $\mathcal{V}_j$ and those of $\mathcal{V}_{j-1}$ is indicated by

$$\phi_{j,0}(t) = \sum_k g_k \phi_{j-1,k}(t).$$

The coefficients g_k of the dilation equation are also the coefficients of a lowpass filter.

In the case of the wavelets functions, there is

$$\psi_{j,0}(t) = \sum_k h_k \phi_{j-1,k}(t);$$

and the coefficients h_k of this dilation equation are the coefficients of a higpass filter.

Conditions Imposed on the Dilation Coeffiients

Various conditions must be imposed on the coefficients of the dilation equations. The first pair of conditions are necessitated by the two-scale dilation relationships:

$$\sum_k g_k = 2^{1/2} \qquad \text{and} \qquad \sum_k h_k = 0.$$

The next conditions are implied by orthonormality of the basis functions:

$$\sum_k g_k^2 = 1 \qquad \text{and} \qquad \sum_k h_k^2 = 1.$$

To ensure the orthogonality of scaling fuctions and wavelets at different displacments, which is *sequential orthogonality,* it is necessary that

$$\sum_k g_k g_{k+2m} = 0 \qquad \text{and} \qquad \sum_k h_k h_{k+2m} = 0.$$

To ensure that the scaling function $\phi(t)$ and the wavelet $\psi(t-m)$ that are at the same or different displacements will be mutually orthogonal, which is *lateral orthogonality,* it is sufficient that

$$\sum_k g_k h_{k+2m} = 0, \qquad \text{for all} \quad m.$$

These various conditions will guarantee the complete orthogonality of the basis functions.

The Filters and their z-Transforms

If the dilation coefficients of the scaling function are $g_0, g_1, \ldots, g_{M-1}$, then the conditions of *lateral* orthogonality will be fulfilled if

$$h_k = (-1)^k g_{M-1-k}, \quad \text{which implies that} \quad g_k = (-1)^{k+1} h_{M-1-k}.$$

An example is provided by the case where $M = 4$. Then, there are

$$\begin{aligned} &g_0, \qquad && h_0 = g_3, \\ &g_1, && h_1 = -g_2, \\ &g_2, && h_2 = g_1, \\ &g_3, && h_3 = -g_0. \end{aligned}$$

The z-transforms of these coefficient sequences are

$$\begin{aligned} G(z) &= g_0 + g_1 z + g_2 z^2 + g_3 z^3 = z^3 H(-z^{-1}) = z^{M-1} H(-z^{-1}), \\ H(z) &= g_3 - g_2 z + g_1 z^2 - g_0 z^3 = -z^3 G(-z^{-1}) = -z^{M-1} G(-z^{-1}). \end{aligned}$$

The autocovariance generating functions formed from these coefficients are

$$P(z) = G(z)G(z^{-1}) \qquad \text{and} \qquad Q(z) = H(z)H(-z^{-1}) = G(z)G(-z^{-1})$$

Conditions Affecting the Filter Polynomials

The conditions of *sequential* orthogonality for the scaling function is that coefficients of $P(z)$ associated with the even powers of z are zeros. The coefficients with even powers are comprised by $P(z) + P(-z)$. Given that $p_0 = 1$, there is

$$\begin{aligned} 2 &= P(z) + P(-z) \\ &= G(z)G(z^{-1}) + G(-z)G(-z^{-1}) \quad && \textit{Lowpass Sequential Orthogonality,} \\ &= G(z)G(z^{-1}) + H(z)H(z^{-1}) && \textit{Power Complementarity,} \\ &= H(z)H(z^{-1}) + H(-z)H(-z^{-1}) && \textit{Highpass Sequential Orthogonality.} \end{aligned}$$

The cross-covariance generating function is $R(z) = G(z)H(z^{-1})$. The condition that wavelets and the scaling functions are mutual orthogonality at even-valued displacements, is

$$R(z) + R(-z) = G(z)H(z^{-1}) + G(-z)H(-z^{-1}) = 0 \quad \textit{Lateral Orthogonality.}$$

The conditions that the elements of the basis elements of $\mathcal{V}_1$ and $\mathcal{W}_1$, are mutually orthogonal, will guarantee the mutually orthogonality of all of the elements of the final basis that reside in different bands.

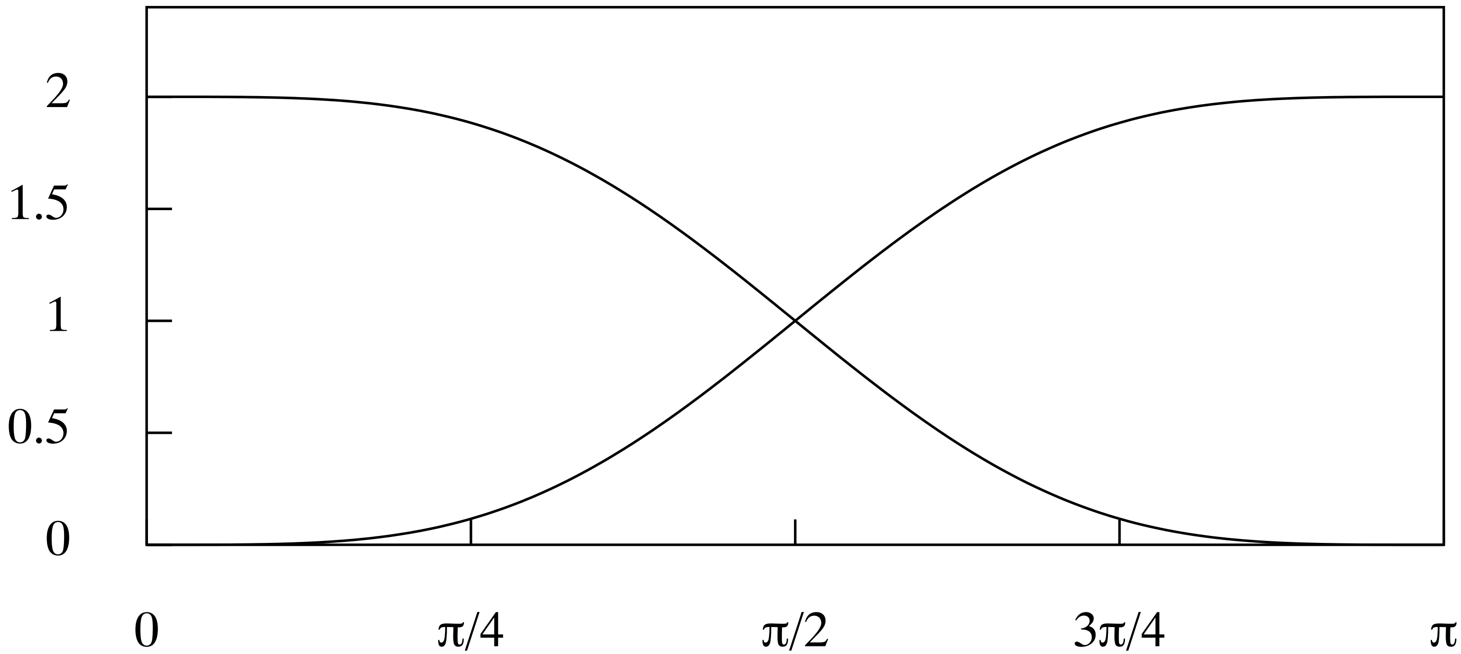

Figure 2. The squared gains of the complementary lowpass and highpass D4 Daubechies filters.

Generating Wavelets and Scaling Functions

In the majority of cases, there are no analytic time-domain representations of the wavelets and the scaling functions. Therefore, iterative procedures, based on the dilation equations, are used in generating graphs of these functions.

The scaling function $\phi(t) \in \mathcal{V}_0$ may be expressed in terms of M scaling functions $\phi(2t-k) \in \mathcal{V}_{-1}; k = 0, 1, \ldots, M-1$ at twice the resolution:

$$\phi(t) = 2^{1/2} \sum_{k=0}^{M-1} g_k \phi(2t-k).$$

These, in turn, may be expressed in terms of scaling functions at the next level:

$$\phi(2t-k) = 2^{1/2} \sum_{j=0}^{M-1} g_j \phi(2[2t-k]-j).$$

The process continues indefinitely until $\phi(t)$ is expressed in terms of an infinite number of coefficients, each associated with scaling function that has become a Dirac delta.

The wavelet function $\psi(t)$ can be expanded in the same way, starting with the equation

$$\psi(t) = 2^{1/2} \sum_{k=0}^{M-1} h_k \phi(2t-k).$$

The Daubechie D4 Wavelet and Scaling Function

An example of a pair of wavelet and scaling functions that have dilation equations of four coefficients is provided by Daubechies' D4 functions. In this case, the lowpass filter coefficient are are

$$\begin{aligned} g_0 &= (1+\sqrt{3})/(4\sqrt{2}), \qquad & g_1 &= (3+\sqrt{3})/(4\sqrt{2}), \\ g_2 &= (3-\sqrt{3})/(4\sqrt{2}), \qquad & g_3 &= (1-\sqrt{3})/(4\sqrt{2}), \end{aligned} \tag{44}$$

The highpass filter coefficients are

$$h_0 = g_3, \quad h_1 = -g_2, \quad h_2 = g_1, \quad h_3 = -g_0.$$

The profiles of the wavelet and scaling functions derived from these coefficients are represented in Figures 3 and 4.

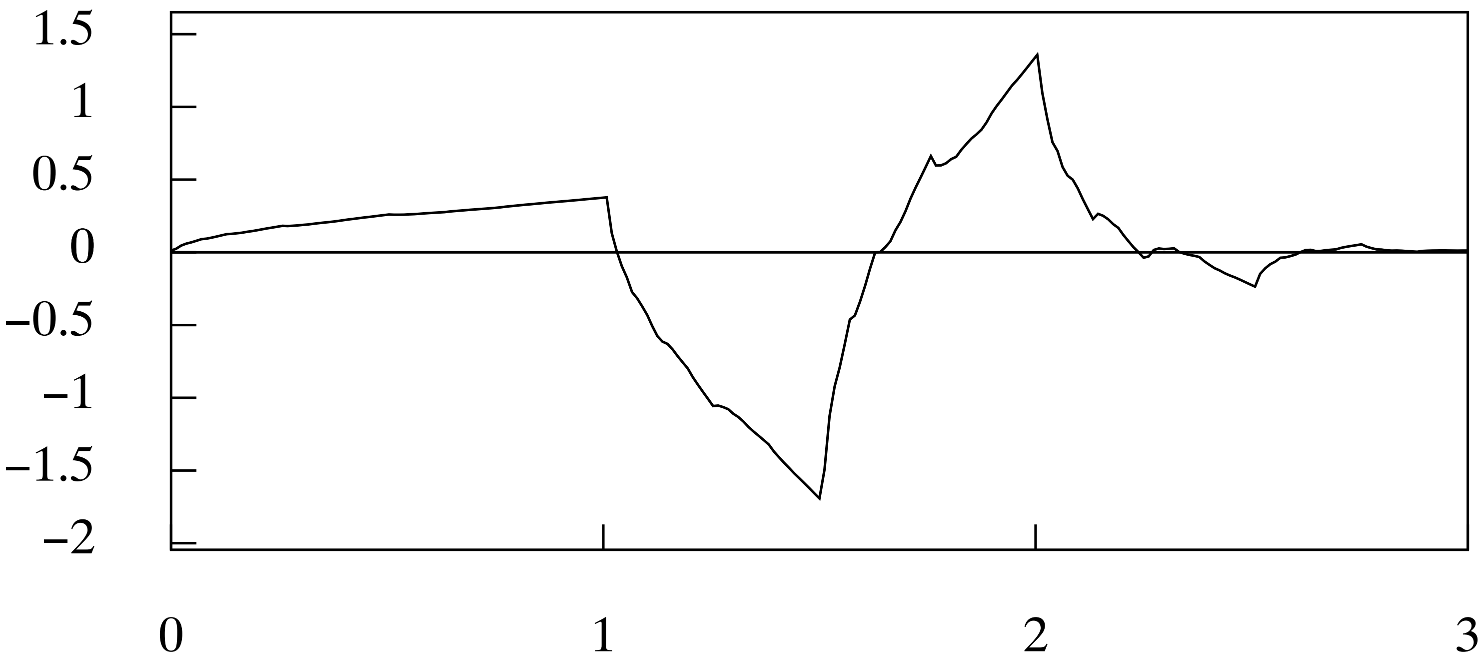

Figure 3. The Daubechies D4 wavelet function calculated via a recursive method.

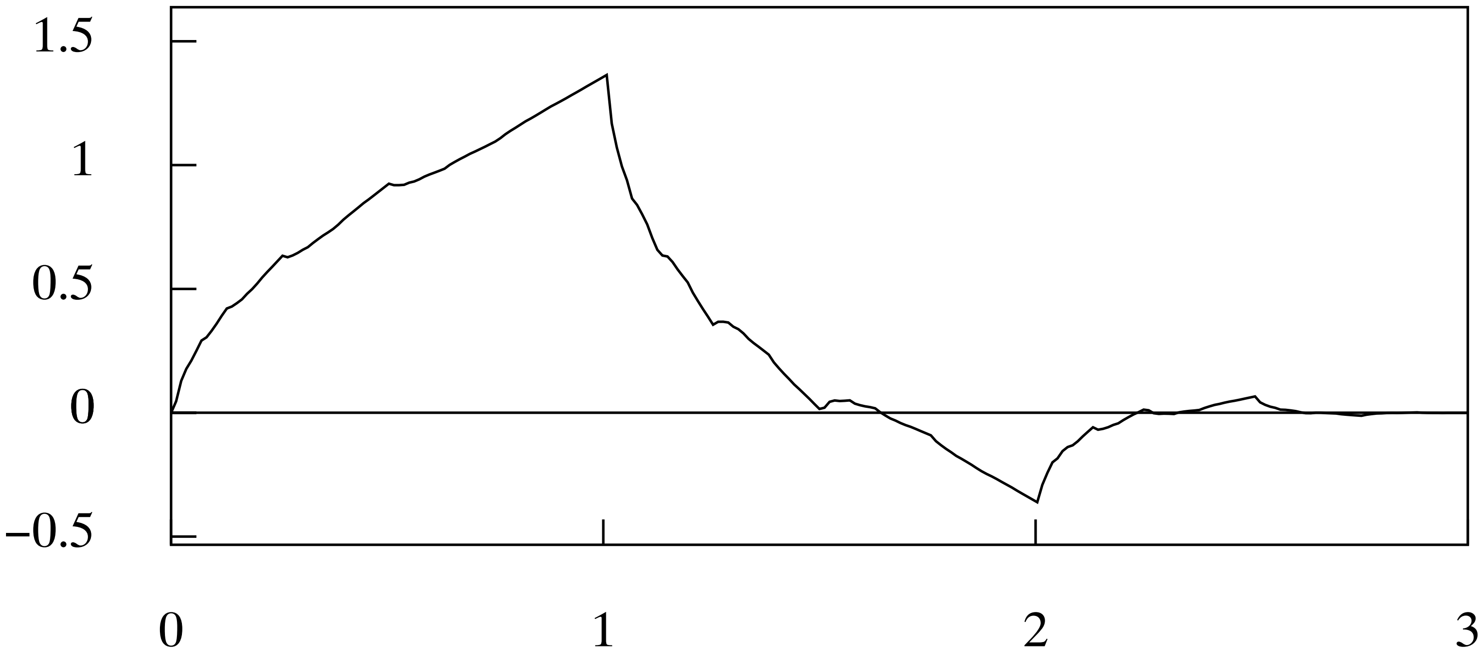

Figure 4. The Daubechies D4 scaling function calculated via a recursive method.

A Matrix Formulation for Wavelets Computations

Matrices are helpful in progamming a wavelets analysis. The downsampling matrix V comes from deleting alternate rows of an identity matrix, beginning with the second row.

An example is the matrix that downsamples a vector of order 6:

$$\mathrm{V}y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_2 \\ y_4 \end{bmatrix}.$$

The corresponding up-sampling matrix $\Lambda = \mathrm{V}'$ has the effect of interpolating zeros between the elements of a vector. An example is the matrix that interpolates zeros into a vector of order 3:

$$\Lambda y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_0 \\ 0 \\ y_1 \\ 0 \\ y_2 \\ 0 \end{bmatrix}.$$

Circulant Filter Matrices

The highpass filter that is applied to the data in the first round of the wavelets decomposition has the following circulant matrix representation:

$$H_{(1)} = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 \\ h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 \end{bmatrix}.$$

This matrix is obtained by replacing the argument z within the polynomial $H(z) = h_0 + h_1 z + h_2 z^2 + h_3 z^3$ by the circulant matrix $K_8 = [e_1, e_2, \ldots, e_7, e_0]$.

Premultiplying $H_{(1)}$ by the down sampling matrix is a matter of deleting alternate rows:

$$M_{(1)} = \mathrm{V}H_{(1)} = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 \end{bmatrix}.$$

Wavelet Analysis and Wavelet Synthesis

Let $y = [y_0, \ldots, y_{T-1}]'$, with $T = 2^n$, be the data vector, and let $\beta = [\beta_0, \ldots, \beta_{T-1}]'$ be the vector of the wavelets amplitude coefficients. Here, $\beta_{T-1} = \gamma_{n0}$ is the coefficient of the single scaling function in the ultimate subdivision of the frequency range.

The mappings from y to β, in the analysis stage, and from β to y in the synthesis stage, are

$$\beta = Q'y \qquad \text{and} \qquad y = Q\beta,$$

where Q ia an orthonormal matrix such that $QQ' = Q'Q = I_T$.

The coefficients of β can be grouped according their level in the dyadic decomposition:

$$\beta = [\beta'_{(1)}, \beta'_{(2)}, \ldots, \beta'_{(n)}, \gamma'_{(n)}]'$$

Then, at the jth level, there is

$$\beta_{(j)} = \mathrm{V}H_{(j)}\mathrm{V}G_{(j-1)} \cdots \mathrm{V}G_{(1)}y = Q'_{(j)}y.$$

The vector $\beta_{(j)}$ is entailed in the synthesis of the vector $w_j = [w_{0j}, w_{1j}, \ldots, w_{T-1,j}]'$ within the decomposition of $y = w_1 + \cdots + w_n + v_n$:

$$w_j = Q_{(j)}\beta_{(j)} = \{G'_{(1)}\Lambda \cdots G'_{(j-1)}\Lambda H'_{(j)}\Lambda\}\beta_{(j)}.$$

The First Stage of Dyadic Decomposition

When $M_{(1)} = \mathrm{V}H_{(1)}$ is combined with the matrix $L_{(1)} = \mathrm{V}G_{(1)}$, which is the down-sampled version of the lowpass filter matrix $G_{(1)} = G(K_8)$, and when the data vector y is mapped through the combined matrix, the result is

$$\begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \hline \gamma_{10} \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{bmatrix} = \left[\begin{array}{cccccccc} h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 \\ \hline g_0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1 \\ g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & g_3 \\ 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0 \end{array}\right] \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix}.$$

The equations can be represented, in summary notation, by

$$\begin{bmatrix} \beta_{(1)} \\ \gamma_{(1)} \end{bmatrix} = \begin{bmatrix} \mathrm{V}H_{(1)} \\ \mathrm{V}G_{(1)} \end{bmatrix} y.$$

This equation will serve to represent the general case, where $T = 2^n$ and $M = 2q$ are unspecified values.

The Lapped Orthogonal Transform

An alternative arrangement of the equations is described as a Lapped Orthogonal Transform (LOT):

$$\left[\begin{array}{c}\beta_{10}\\ \gamma_{10}\\ \hline \beta_{11}\\ \gamma_{11}\\ \hline \beta_{12}\\ \gamma_{12}\\ \hline \beta_{13}\\ \gamma_{13}\end{array}\right]=\left[\begin{array}{cccccccc}h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1\\ g_0 & 0 & 0 & 0 & 0 & g_3 & g_2 & g_1\\ \hline h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & h_3\\ g_2 & g_1 & g_0 & 0 & 0 & 0 & 0 & g_3\\ \hline 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0\\ 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0\\ 0 & 0 & 0 & g_3 & g_2 & g_1 & g_0 & 0\end{array}\right]\left[\begin{array}{c}y_0\\ y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\\ y_7\end{array}\right].$$

We shall have occasion to mention this arrangement at a later stage when we deal with the transform coding that is used commonly in connection with digital images.

The Second Stage of Dyadic Decomposition

In the second round of the wavelets decomposition, the coefficients associated with the level-1 wavelets are preserved and the coefficients associated with the level-1 scaling functions are subject to a further decomposition:

$$\begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \hline \beta_{20} \\ \beta_{21} \\ \hline \gamma_{20} \\ \gamma_{21} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & h_0 & h_3 & h_2 & h_1 \\ 0 & 0 & 0 & 0 & h_2 & h_1 & h_0 & h_3 \\ \hline 0 & 0 & 0 & 0 & g_0 & g_3 & g_2 & g_1 \\ 0 & 0 & 0 & 0 & g_2 & g_1 & g_0 & g_3 \end{array}\right] \begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \hline \gamma_{10} \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{bmatrix}.$$

The summary notation for this is

$$\begin{bmatrix} \beta_{(1)} \\ \beta_{(2)} \\ \gamma_{(2)} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathrm{V}H_{(2)} \\ 0 & \mathrm{V}G_{(2)} \end{bmatrix} \begin{bmatrix} \beta_{(1)} \\ \gamma_{(1)} \end{bmatrix}.$$

The Final Stage of Dyadic Decomposition

In the next round of filtering, which is the final stage in this example, there are only two data points to be mapped through the filters.

The consequence is that γ_{20} and γ_{21} must be used twice in the third and final transformation. This can represented equally by

$$\begin{bmatrix} \gamma_{30} \\ \beta_{30} \end{bmatrix} = \begin{bmatrix} h_0 & h_3 & h_2 & h_1 \\ g_0 & g_3 & g_2 & g_1 \end{bmatrix} \begin{bmatrix} \gamma_{20} \\ \gamma_{21} \\ \gamma_{20} \\ \gamma_{21} \end{bmatrix},$$

or by

$$\begin{bmatrix} \gamma_{30} \\ \beta_{30} \end{bmatrix} = \begin{bmatrix} h_0 + h_2 & h_3 + h_1 \\ g_0 + g_2 & g_3 + g_1 \end{bmatrix} \begin{bmatrix} \gamma_{20} \\ \gamma_{21} \end{bmatrix}.$$

On the LHS is a vector containing the amplitude coefficients, respectively, of a wavelet and a scaling function stretching the length of the data sequence.

The Final Stage of a Wavelets Synthesis

The conditions of sequential and lateral othogonality imply that the first-stage analysis matrix is orthonomal, with a transpose that is its inverse. Therefore,

$$\begin{bmatrix} H'_{(1)}\Lambda & G'_{(1)}\Lambda \end{bmatrix} \begin{bmatrix} \mathrm{V}H_{(1)} \\ \mathrm{V}G_{(1)} \end{bmatrix} = I.$$

It follows that the data vector y can be recovered from $\gamma_{(1)}$ and $\beta_{(1)}$ via

$$y = G'_{(1)}\Lambda\gamma_{(1)} + H'_{(1)}\Lambda\beta_{(1)}.$$

In terms of the example with $T = 8$ and $M = 4$, this can be rendered as

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} g_0 & g_2 & 0 & 0 \\ 0 & g_1 & g_3 & 0 \\ 0 & g_0 & g_2 & 0 \\ 0 & 0 & g_1 & g_3 \\ 0 & 0 & g_0 & g_2 \\ g_3 & 0 & 0 & g_1 \\ g_2 & 0 & 0 & g_0 \\ g_1 & g_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{10} \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{bmatrix} + \begin{bmatrix} h_0 & h_2 & 0 & 0 \\ 0 & h_1 & h_3 & 0 \\ 0 & h_0 & h_2 & 0 \\ 0 & 0 & h_1 & h_3 \\ 0 & 0 & h_0 & h_2 \\ h_3 & 0 & 0 & h_1 \\ h_2 & 0 & 0 & h_0 \\ h_1 & h_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix}.$$

The Two-Phase Representation

The two-phase formulation separates the data sequence into elements with even-valued indices and elements with odd-valued indices. It is more efficient in real-time processing.

The first stage of the analysis transformation is as follows:

$$
\begin{bmatrix} \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \\ \hline \gamma_{10} \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \end{bmatrix}
=
\begin{bmatrix} h_0 & 0 & 0 & h_2 \\ h_2 & h_0 & 0 & 0 \\ 0 & h_2 & h_0 & 0 \\ 0 & 0 & h_2 & h_0 \\ \hline g_0 & 0 & 0 & g_2 \\ g_2 & g_0 & 0 & 0 \\ 0 & g_2 & g_0 & 0 \\ 0 & 0 & g_2 & g_0 \end{bmatrix}
\begin{bmatrix} y_0 \\ y_2 \\ y_4 \\ y_6 \end{bmatrix}
+
\begin{bmatrix} 0 & 0 & h_3 & h_1 \\ h_1 & 0 & 0 & h_3 \\ h_3 & h_1 & 0 & 0 \\ 0 & h_3 & h_1 & 0 \\ \hline 0 & 0 & g_3 & g_1 \\ g_1 & 0 & 0 & g_3 \\ g_3 & g_1 & 0 & 0 \\ 0 & g_3 & g_1 & 0 \end{bmatrix}
\begin{bmatrix} y_1 \\ y_3 \\ y_5 \\ y_7 \end{bmatrix}.
$$

Omitting the subscripts that indicate the level of the decomposition, this can be represented, in summary notation, by

$$
\begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} H^e \\ G^e \end{bmatrix} y^e + \begin{bmatrix} KH^o \\ KG^o \end{bmatrix} y^o,
$$

where $K = [e_1, e_2, \ldots, e_{T-1}, e_0]$ is the circulant lag operator.

The Final Stage of the Two-Phase Synthesis

An alternative way of formatting the final synthesis equation, which delivers the two phases of the data, is

$$\begin{bmatrix} y_0 \\ y_2 \\ y_4 \\ y_6 \\ \hline y_1 \\ y_3 \\ y_5 \\ y_7 \end{bmatrix} = \left[\begin{array}{cccc} h_0 & h_2 & 0 & 0 \\ 0 & h_0 & h_2 & 0 \\ 0 & 0 & h_0 & h_2 \\ h_2 & 0 & 0 & h_0 \\ \hline 0 & h_1 & h_3 & 0 \\ 0 & 0 & h_1 & h_3 \\ h_3 & 0 & 0 & h_1 \\ h_1 & h_3 & 0 & 0 \end{array}\right] \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \left[\begin{array}{cccc} g_0 & g_2 & 0 & 0 \\ 0 & g_0 & g_2 & 0 \\ 0 & 0 & g_0 & g_2 \\ g_2 & 0 & 0 & g_0 \\ \hline 0 & g_1 & g_3 & 0 \\ 0 & 0 & g_1 & g_3 \\ g_3 & 0 & 0 & g_1 \\ g_1 & g_3 & 0 & 0 \end{array}\right] \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

In the summary notation, this becomes

$$\begin{bmatrix} y^e \\ y^o \end{bmatrix} = \begin{bmatrix} H^{e\prime} \\ H^{o\prime}K' \end{bmatrix}\beta + \begin{bmatrix} G^{e\prime} \\ G^{o\prime}K' \end{bmatrix}\gamma.$$

The Maximal Overlap Discrete Wavelet Transform (MODWT)

A Maximal Overlap Discrete Wavelet Transform (MODWT) arises from a dyadic Discrete Wavelet Transform (DWT) when the operation of down sampling, which accompanies each stage of the pyramid algorithm, is omitted.

In place of the $T/2^j$ wavelet amplitude coefficients arising from the jth stage of the DWT, the MODWT generates a full vector β_j of T coefficients. Morever, there is no longer any purpose in the restriction that $T = 2^n$.

In contrast to those generated by a DWT, the MODWT components are not mutually orthogonal. However, their sum does equal the original data vector.

Whatever is achieved by a MODWT can be acheived by a battery of complementary bandpass filters, with a greater feedom of choice amongst alternative filter specifications.