TWO-CHANNEL FILTER BANKS

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In this lecture, we will describe the structure of the two-channel filter bank, which is the basis of a dyadic wavelets analysis.

Once the architecture of the two-channel filter bank has been established, it can be used in pursuit of a dyadic decomposition of the f requency range in which, in descending the frequency scale, successive frequency bands have half the width of their predecessors. Thereafter, we may consider dividing the frequency range into 2^n bands of equal width.

The wavelets form a basis in continuous time that is appropriate to the description of a frequency-limited function, According to the sampling theorem, such functions can be represented by discrete-time sequences, which will be the subject of the present lecture.

The continuous-time wavelets might be regarded as a shadowy accompaniment —and even an inessential one—of a discrete-time wavelet analysis that can be recognised as an application of the techniques of multirate fitering that are nowadays prevalent in communications engineering.

Subband Coding and Transmission

The signal is split between two frequency bands via highpass and lowpass filters. Then it is subsampled and encoded for transmission. At the receiving end, the signals of the two channels are decoded and upsampled, by the interpolation of zeros between the sample elements. They are smoothed by filtering, and recombined to provide a reconstuction of the original signal.

The path taken by the signal through the highpass branch of the two-channel network may be denoted by

$$y(z) \longrightarrow H(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow E(z) \longrightarrow w(z),$$

whereas the path taken through the lowpass branch may be denoted by

$$y(z) \longrightarrow G(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow D(z) \longrightarrow v(z).$$

Here, $G(z)$ and $H(z)$ are the analysis filters, and $D(z)$ and $E(z)$ are the synthesis filters.

Downsampling, which deletes signal elements with odd-values indices, is denoted by $(\downarrow 2)$. Upsampling, which inserts zeros between adjacent elements, is denoted by $(\uparrow 2)$.

The output signal, formed by merging the two branches, is $x(t) = v(t) + w(t)$.

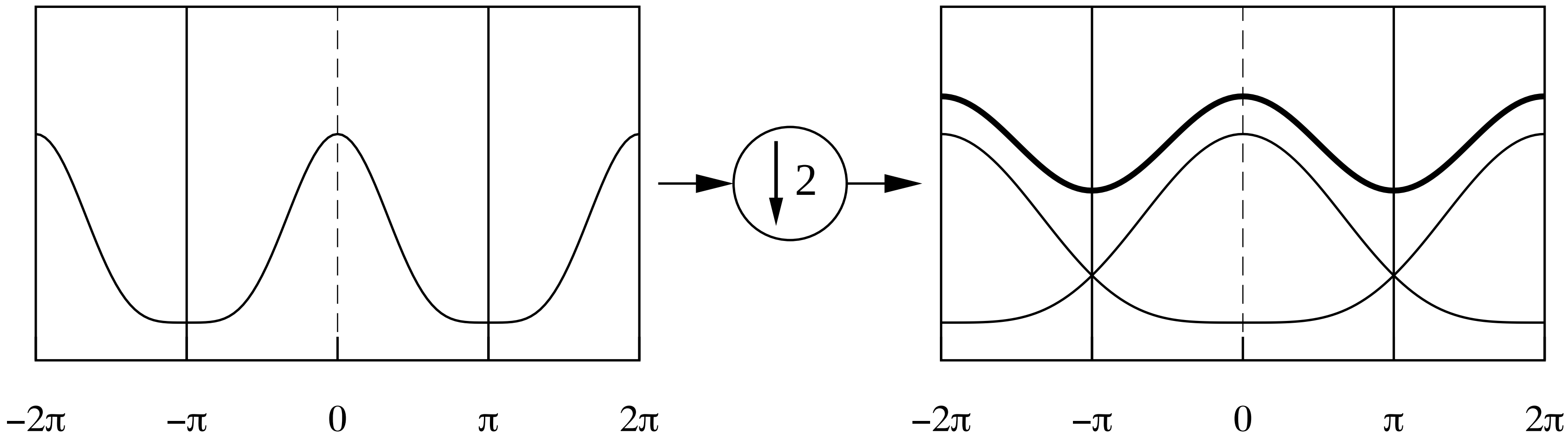

Figure 1. The effects in the frequency-domain of downsampling by a factor of 2. On the left is the original spectral function and on the right is its dilated version together with a copy displaced by 2π radians. To show the effects of aliasing, the ordinates of the two functions must be added.

Spectral Effects of Downsampling and Upsampling

Let $p(t) \longleftrightarrow p(\omega)$ be an absolutely summable sequence and its Fourier transform. In downsampling, ω is replaced by $\omega/2$. Therefore,

$$p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(\omega/2) + p(\pi + \omega/2).$$

Define $z = \exp\{-\mathrm{i}\omega\}$ and observe that $\exp\{\pm\mathrm{i}\pi\} = -1$ and $\exp\{-\mathrm{i}(\pi + \omega/2)\} = -\exp\{-\mathrm{i}\omega/2\}$. Then,

$$p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(z^{1/2}) + p(-z^{1/2})\}.$$

Through upsampling, ω is multiplied by 2 and the frequency function undergoes two cycles as ω traverses an interval of 2π. Therefore,

$$p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(\omega) + p(\pi + \omega)\},$$

which can also be written as

$$p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(z) + p(-z)\}.$$

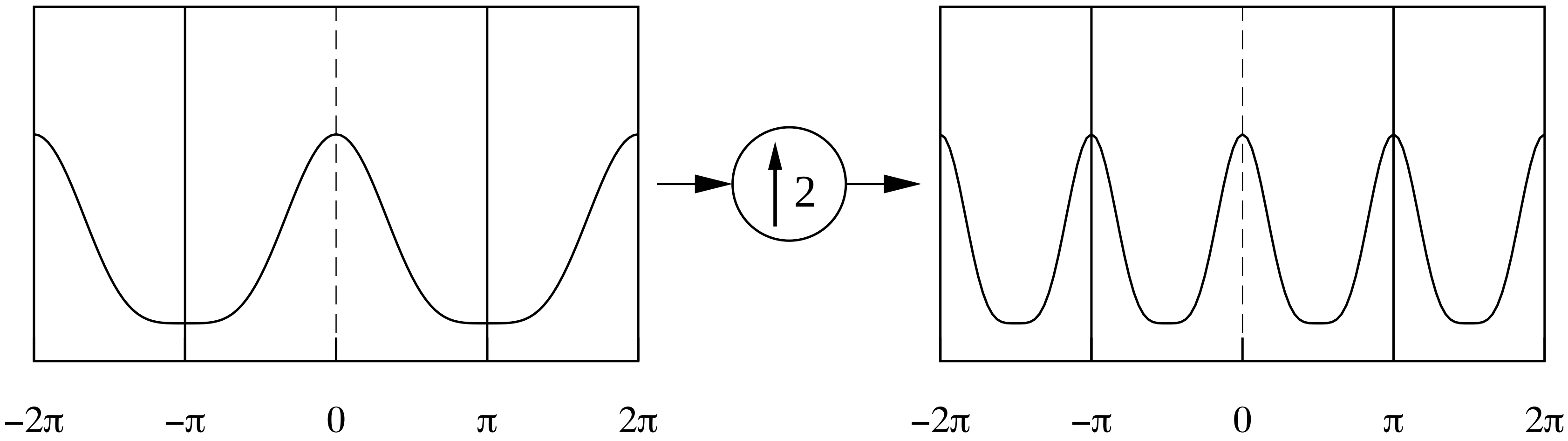

Figure 2. The effects in the frequency-domain of upsampling by a factor of 2. On the left is the original spectral function. On the right, the rate at which the frequency function evolves has been doubled and the function undergoes two cycles as ω traverses an interval of 2π radians .

Perfect Reconstruction within the Two-Channed Network

The signals that emerge from the two channels are given by

$$w(z) = \frac{1}{2}E(z)\{H(z)y(z) + H(-z)y(-z)\},$$

$$v(z) = \frac{1}{2}D(z)\{G(z)y(z) + G(-z)y(-z)\}.$$

Adding the two signals gives

$$\begin{aligned} x(z) = &\frac{1}{2}\{D(z)G(-z) + E(z)H(-z)\}y(-z), \\ &+ \frac{1}{2}\{D(z)G(z) + E(z)H(z)\}y(z). \end{aligned}$$

The term in $y(-z)$ represents the aliasing effect. The term in $y(z)$, disregarding the factor of 1/2, would represent the effect of the network without upsampling and down-sampling.

Perfect reconstruction of the signal is achieved if

$$D(z)G(z) + E(z)H(z) = 2, \qquad \text{(i)}$$

$$D(z)G(-z) + E(z)H(-z) = 0. \qquad \text{(ii)}$$

The Network in the Absense of an Aliasing Effect

For an illustration, assume that $G(z) = G(z^{-1})$ and $H(z) = H(z^{-1})$ are the ideal and symmetric halfband filters, for which $H(z) = G(-z)$, $G(z) = H(-z)$ and $G(z^{-1})H(z) = 0$.

Then, the filters are mirror images of each other, they are mutually orthogonal, and they have no spectral overlap.

The appropriate choice of the synthesis filters would be the reversed filters $D(z) = G(z^{-1})$ and $E(z) = H(z^{-1})$ which, in view of the symmetry, are none other than $G(z)$and $H(z)$ respectively.

The following diagrams show the effects of the filtering and of the downsampling and upsampling in the two channels in the absence of an aliasing effect, and they show how the original signal would be reconstructed perfectly as the sum of the outputs of the two channels.

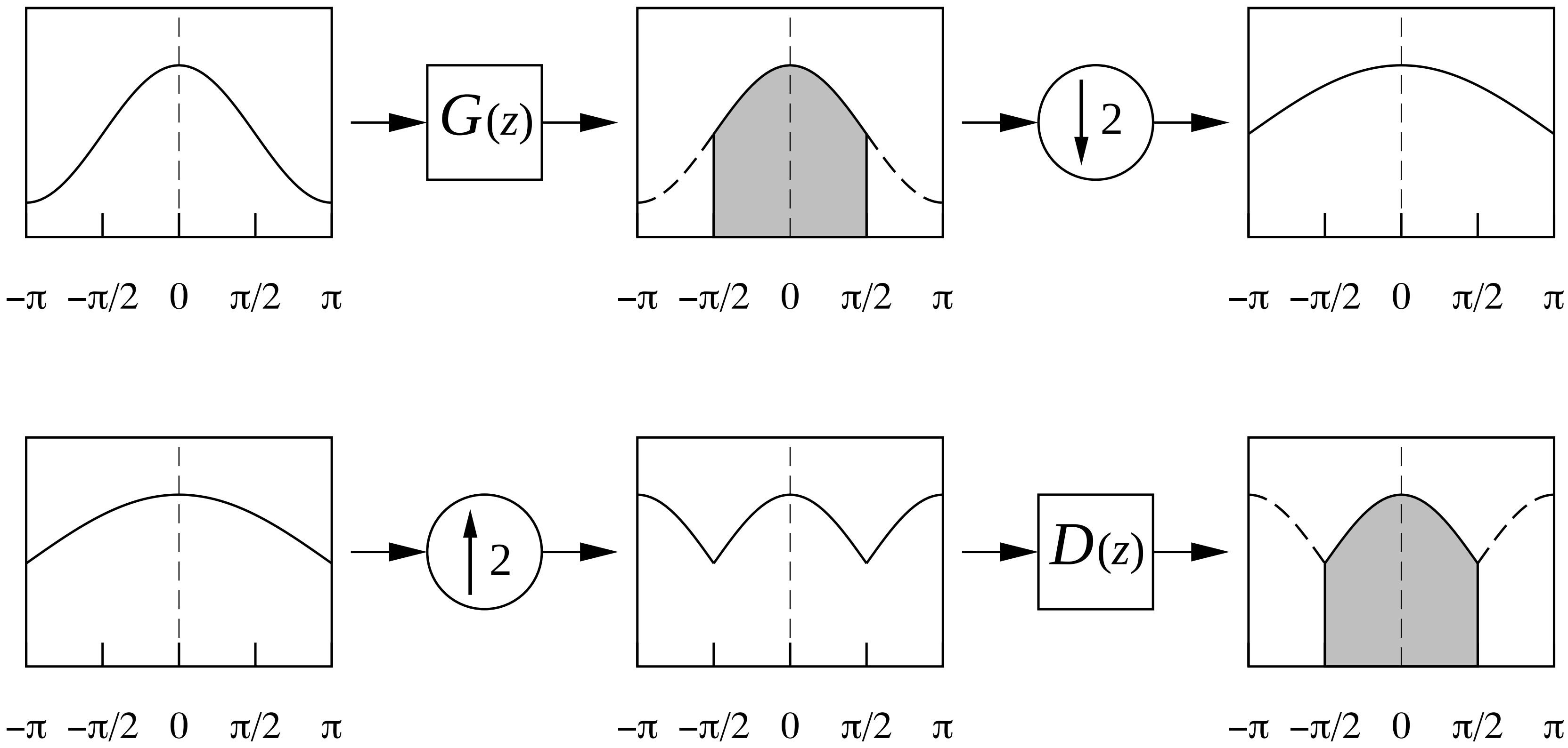

Figure 3. The effects on the spectrum of the lowpass branch of the two-channel filter bank, in the case where $G(z)$ and $D(z) = G(z)$ are ideal halfband filters.

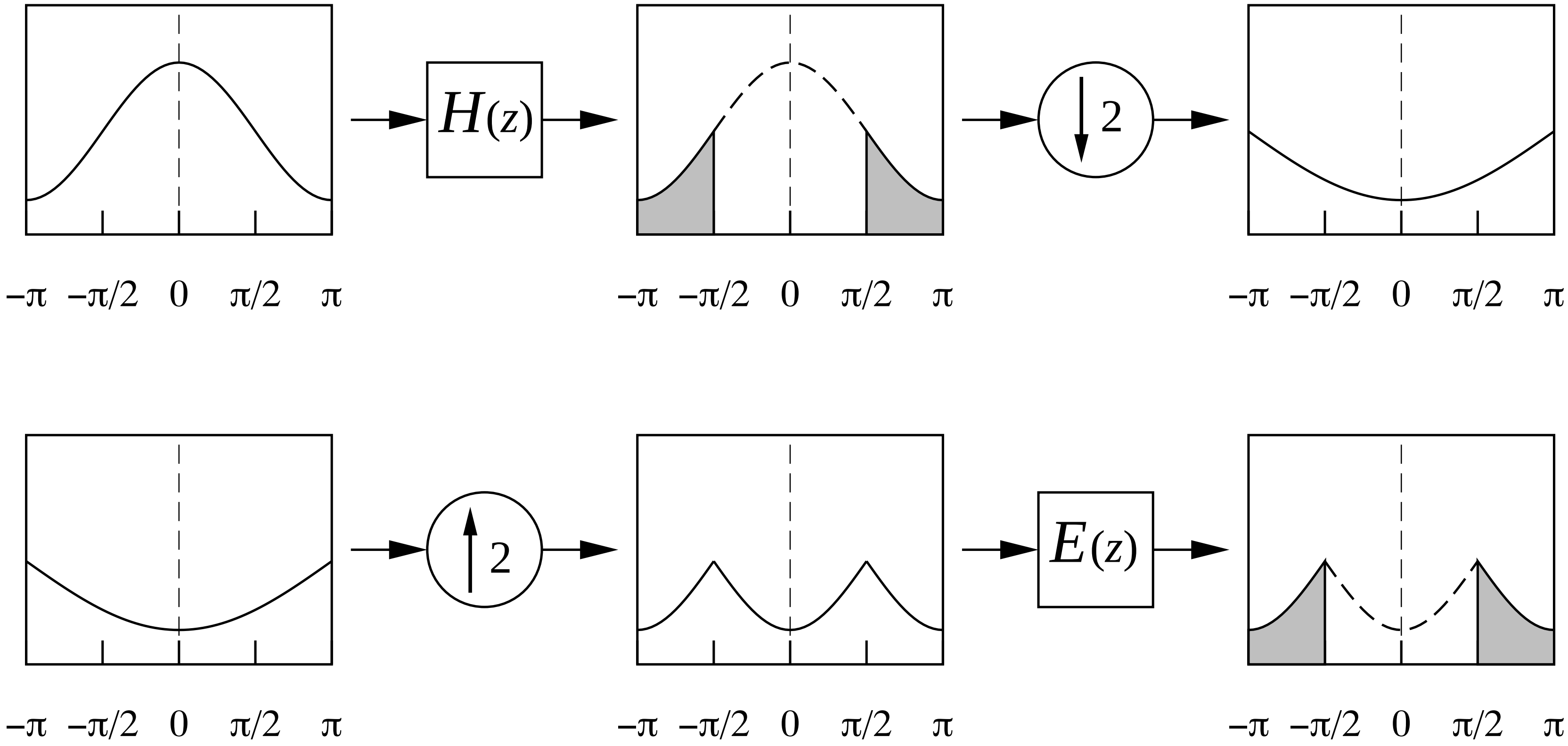

Figure 4. The effects on the spectrum of the highpass branch of the two-channel filter bank, in the case where $H(z)$ and $E(z) = H(z)$ are ideal halfband filters.

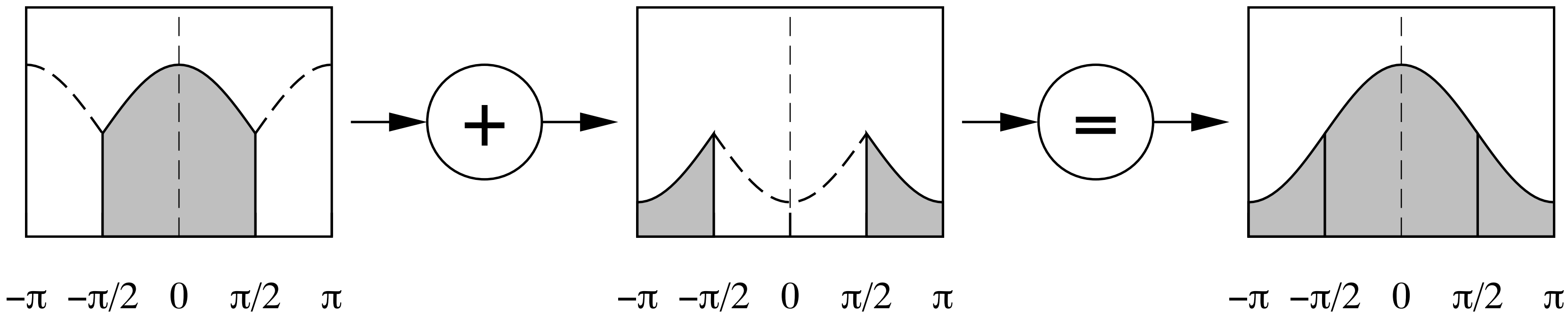

Figure 5. The perfect reconstruction of the signal spectrum via the addition of the lowpass spectrum and the highpasss spectrum, in the case where the two-channel filter bank incorporates the ideal halfband filters.

Conditions for Perfect Reconstruction

Given the specifications of the analysis filters $G(z)$ and $H(z)$, it is necessary to solve the conditions of perfect reconstruction to find the specification of the synthesis filters $D(z)$ and $E(z)$.

The equation to be solved is

$$[\,D(z) \quad E(z)\,]\begin{bmatrix} G(z) & G(-z) \\ H(z) & H(-z) \end{bmatrix} = [\,2 \quad 0\,].$$

This gives

$$\begin{aligned}\begin{bmatrix} D(z) \\ E(z) \end{bmatrix} &= \begin{bmatrix} G(z) & H(z) \\ G(-z) & H(-z) \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{2}{\Delta(z)} \begin{bmatrix} H(-z) & -H(z) \\ -G(-z) & G(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix},\end{aligned}$$

where

$$\Delta(z) = G(z)H(-z) - H(z)G(-z) = -\Delta(-z)$$

is a determinant. Thus

$$D(z) = \frac{2}{\Delta(z)}H(-z) \qquad \text{and} \qquad E(z) = \frac{-2}{\Delta(z)}G(-z).$$

Quadrature Mirror Filters

Esterban and Galand (1977) proposed to use so-called quadrature mirror filters, of which the frequency response of one filter is the mirror image, about the value $\pi/2$, of that of the other. Thus $H(z) = G(-z)$. The additional specifications were

$$D(z) = G(z) \quad \text{and} \quad E(z) = -H(z) = -G(-z).$$

The condition for the cancellation of the aliasing effects arising from the downsampling, which is that $D(z)G(-z) + E(z)H(-z) = 0$, is satisfied, since it becomes $G(z)G(-z) + \{-G(-z)\}\{G(z)\} = 0$.

The other condition of perfect reconstruction, which, with the allowance of a lag, requires that $D(z)G(z) + E(z)H(z) = 2z^q$, for some integer lag value q, will be satisfied if the filters can be chosen such that

$$G(z)G(z) - G(-z)G(-z) = G(z)G(z) - H(z)H(z) = 2z^q.$$

This is hard to achieve. In fact, for FIR filters, the condition cannot be satisfied exactly except by the Haar filter. For the causal Haar filter $G(z) = (1 + z)/\sqrt{2}$, the condition becomes

$$\frac{1}{2}(1 + 2z + 2z^2) - (1 - 2z + 2z^2) = 2z.$$

Conditions of Orthogonality

The additional restrictions of orthogonality are liabel to be imposed. These affect the autocorrelation functions of the filters, which are $G(z)G(z^{-1})$, for the highpass filter, and $H(z)H(z^{-1})$, for the lowpass filter, and the cross-correlation function $G(z)H(z^{-1})$.

The sequential orthogonality at displacements that are multiples of two points implies that the downsampled functions must be zero-valued, except for the nonzero central coefficients, to which a value of 2 may be assigned. Thus

$$G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2 \qquad \textit{Lowpass Sequential Orthogonality}$$

and

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2. \qquad \textit{Highpass Sequential Orthogonality}$$

The conditions of lateral orthogonality are expressed in terms of the downsampled version of the cross correlation function of the filters:

$$G(z)H(z^{-1}) + G(-z)H(-z^{-1}) = 0. \qquad \textit{Lateral Orthogonality}$$

Filters that obey the conditions of orthogonality in addition to those of perfect reconstruction may be described as *canonical* filters.

Time-Domain Orthogonal Filters: The Analysis Filters

Smith and Barnwell (1984) solved the problem of achieving both perfect reconstruction and orthogonality with FIR filters with an even number of coeffcients. Since they cannot be symmetric about a central point, these filters will have a nonlinear phase effect.

To see the necessity of an even number of coefficients, consider $G(z) = g_0 + g_1 z + \cdots + g_{M-1}z^{M-1}$ in the case where M is an odd number, as it must be for central symmetry. Then, if $2n = M - 1$, there is $p_{2n} = g_0 g_{M-1} \neq 0$, since $g_0, g_{M-1} \neq 0$, by definition. Therefore, the condition of sequential orthogonality is violated.

According to the prescription of Smith and Barnwell, the lowpass and highpass filters of the analysis section should bear the following relationship:

$$G(z) = z^{M-1}H(-z^{-1}) \qquad \text{or, equivalently,} \qquad H(z) = -z^{M-1}G(-z^{-1}).$$

Observe that, if $H(z) = -G(-z^{-1})$, then the highpass filter would be anti-causal. With $H(z) = -z^{M-1}G(-z^{-1})$, it becomes causal.

Time-Domain Orthogonal Filters: The Synthesis Filters

Using the definitions of $G(z)$ and $H(z)$, and the conditions of sequential orthogonality, the determinant of is evaluated as

$$\begin{aligned}\Delta(z) &= G(z)H(-z) - H(z)G(-z)\\ &= z^{M-1}\left\{G(z)G(z^{-1}) + G(-z)G(-z^{-1})\right\} = 2z^{M-1}\\ &= z^{M-1}\left\{H(z)H(z^{-1}) + H(-z)H(-z^{-1})\right\} = 2z^{M-1}.\end{aligned}$$

In consequence of the conditions of perfect reconstruction, the synthesis filters are

$$D(z) = \frac{2}{\Delta(z)}H(-z) = G(z^{-1}) \quad \text{and} \quad E(z) = \frac{-2}{\Delta(z)}G(-z) = H(z^{-1}),$$

which are the time-reversed versions of the corresponding analysis filters.

Using these, the condition (i) of perfect reconstuction can be expressed as

$$\begin{aligned}2 &= D(z)G(z) + E(z)H(z)\\ &= G(z^{-1})G(z) + H(z)H(z^{-1}).\end{aligned}$$

The final equality is the condition of *Power Complementarity.*

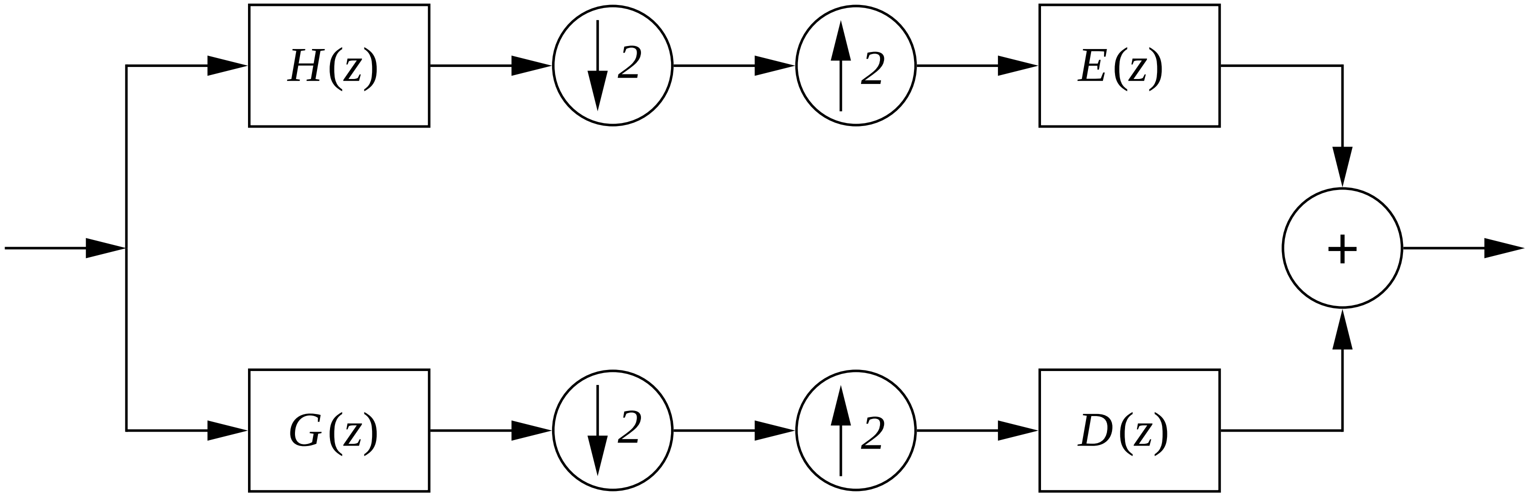

Figure 6. A depiction of the two-channel filter bank. If $H(z) = -z^{M-1}G(-z^{-1})$, then perfect reconstruction and orthogonality can be achieved by setting $E(z) = H(z^{-1})$ and $D(z) = G(z^{-1})$.

Orthogonal Filters in the Frequency Domain

It is possible to achieve perfect reconstruction and orthogonality with symmetric filters that have an infinite number of coefficients. Such filters are bound to have finite supports in the frequency domain. Therefore, it is appropriate to implement them in the frequency domain.

Assume that $G(z) = G(z^{-1})$ is a symmetric half-band filter that satisfies the condition $G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2$ of sequential orthogonality. We may specify that

$$H(z) = -z^{-1}G(-z) \qquad \text{or, equivalently,} \qquad H(-z) = z^{-1}G(z),$$

which implies that

$$G(z) = zH(-z) \qquad \text{or, equivalently,} \qquad G(-z) = -zH(z).$$

The synthesis filters will be given by

$$D(z) = G(z) \qquad \text{and} \qquad E(z) = -zG(-z).$$

Thus, the lowpass synthesis filter is the same as the lowpass analysis filter. The one-point displacement associated with the highpass filter in the analysis section is followed by a one-point displacement in the opposite direction within the synthesis section.

The Gain of the Canonical Filters

In the diagram labelled (a), the envelope centred on the zero frequency and bounded by a continuous line represents the squared gain of a half-band lowpass filter $G(z)$, which is $G(z)G(z^{-1})$. The squared gain $H(z)H(z^{-1})$ of the complementary highpass filter is represented by a broken line. The roles are reversed in diagram (c).

The filters obey the conditions $G(z)G(z^{-1}) + H(z)H(z^{-1}) = 2$ of power complementarity, such that the sum of their squared gains is a constant function with a value of 2.

On the RHS of the diagrams are the squared gains of the filters, as they would be in the case of downsampled data. The downsampling dilates the squared gain functions, extending them over an interval of length 4π, which is wrapped around a circle of circumference 2π to produce diagrams (b) and (d), which are identical.

The aliasing gives rise to the shaded regions of the diagrams on the LHS. In diagram (a), the effects are due to the filter product $D(z)G(-z) = G(z^{-1})G(-z)$. In diagram (c), they are due to the product $E(z)H(-z) = H(z^{-1})H(-z) = -G(z^{-1})G(-z)$. Evidently, these two effects will cancel.

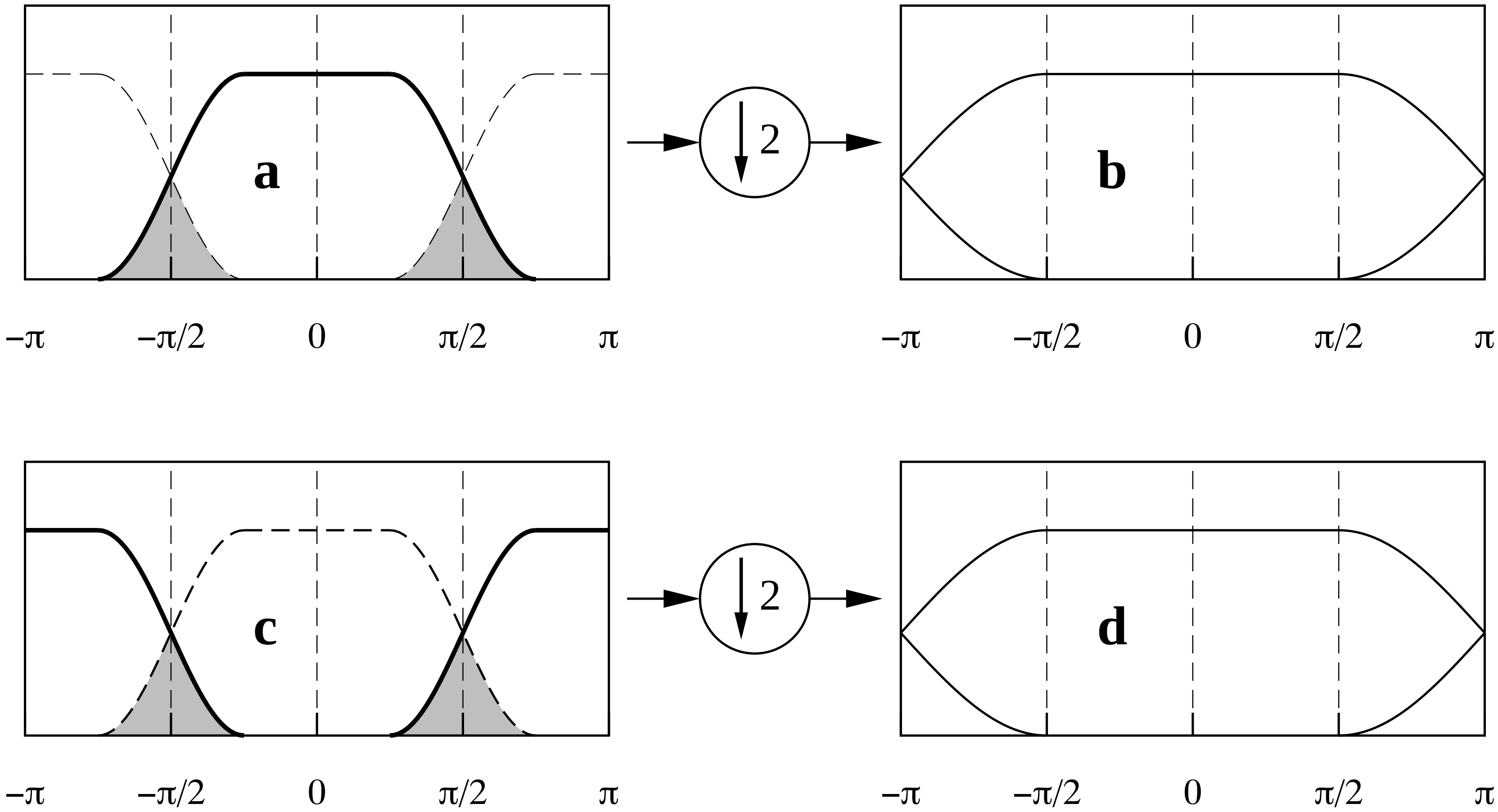

Figure 7. The effects downsampling on the squared gain of the lowpass filter $G(z)$ (top) and on the highpass filter $H(z)$ (bottom).

Processing in Two Phases

A more efficient implementation of the network can be achieved by reversing the order of the filters and the downsampling and upsampling opertions. The latter can be used for splitting the input sequence into its odd and even phases. Thus,

$$y^e(z^2) = \frac{1}{2}\{y(z) + y(-z)\} = \cdots + y_0 + y_2z^2 + y_4z^4 + y_6z^6 + \cdots,$$
$$zy^o(z^2) = \frac{1}{2}\{y(z) - y(-z)\} = \cdots + y_1z + y_3z^3 + y_5z^5 + y_5z^7 + \cdots;$$

which can be added to recover the original sequence:

$$y(z) = y^e(z^2) + zy^o(z^2).$$

The transfer functions of the filters can be partitioned in an analogous manner. Thus

$$G^e(z^2) = \frac{1}{2}\{G(z) + G(-z)\},$$

$$zG^o(z^2) = \frac{1}{2}\{G(z) - G(-z)\}$$

are the components of the lowpass filter

$$G(z) = G^e(z^2) + zG^o(z^2).$$

The highpass filter can be expressed likewise in terms of its components.

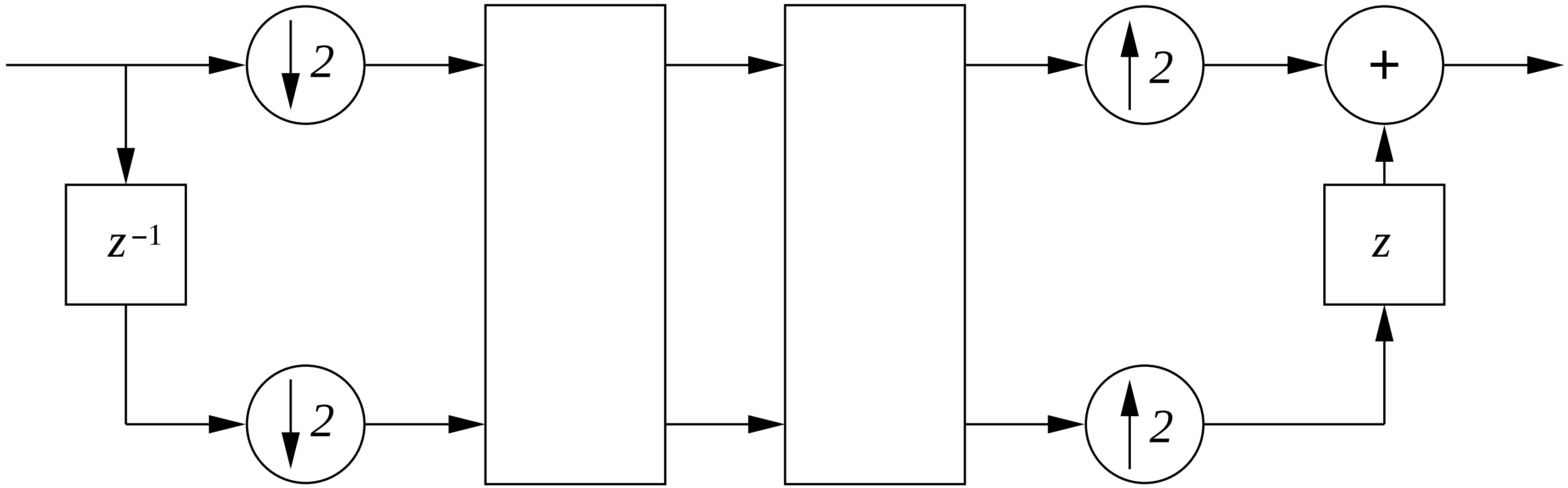

Figure 8. An alternative architecture for the two-channel filter bank separates the data points bearing even indices from those bearing odd indices.

The Equivalence of the Direct Approach and Two-Phase Approach

The even-indexed lowpass signal from downsampling the transform $G(z)y(z)$ is

$$\gamma(z^2) = \frac{1}{2}\{G(z)y(z) + G(-z)y(-z)\} = G^e(z^2)y^e(z^2) + z^2G^o(z^2)y^o(z^2).$$

The analogous highpass signal is

$$\beta(z^2) = \frac{1}{2}\{H(z)y(z) + H(-z)y(-z)\} = H^e(z^2)y^e(z^2) + z^2H^o(z^2)y^o(z^2).$$

These are also the signals that emerge, at the analysis stage, from the two channels of the direct procedure. In terms of the z-transforms, the output of the synthesis transform is

$$y^e(z^2) = H^e(z^{-2})\beta(z) + G^e(z^{-2})\gamma(z^2),$$

$$y^o(z^2) = H^o(z^{-2})\beta(z^2) + G^o(z^{-2})\gamma(z^2).$$

These can be recombined to give

$$y(z) = y^e(z^2) + zy^o(z^2).$$

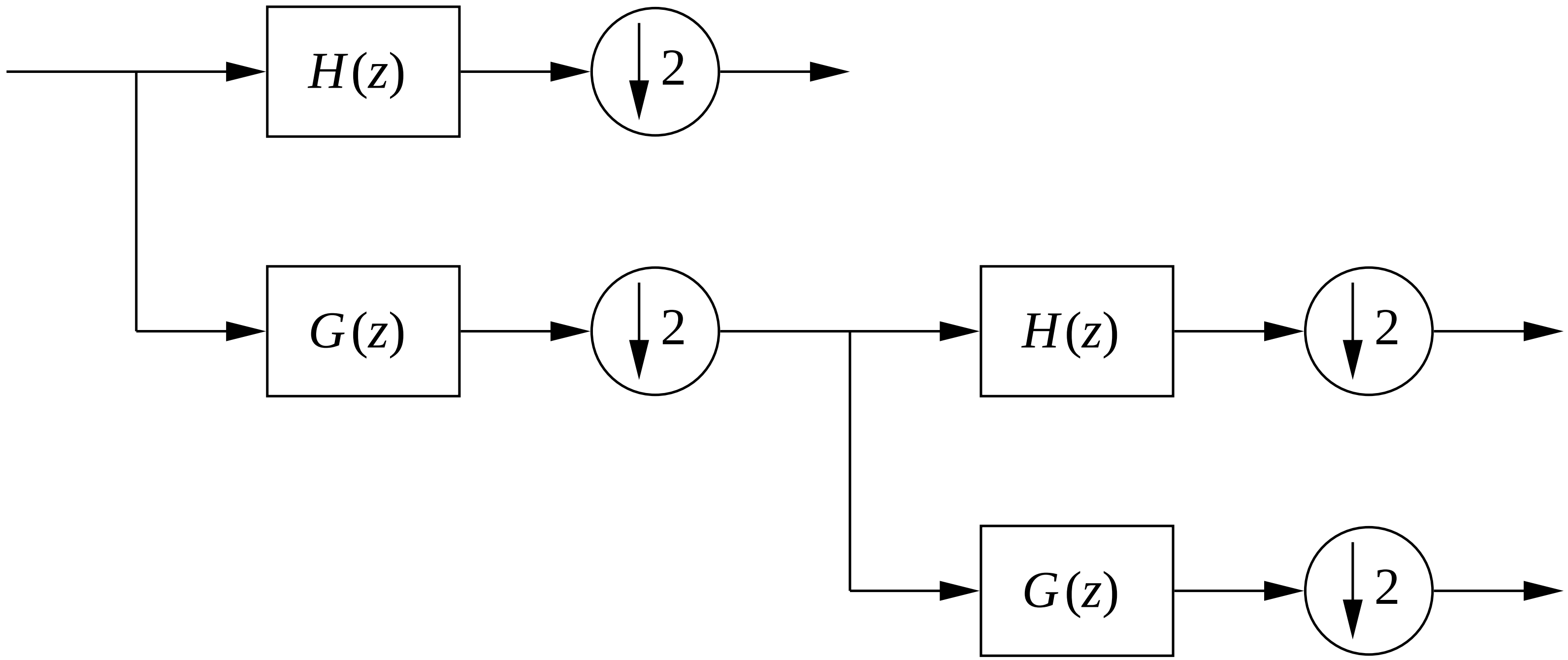

Figure 9. The analysis section of a dyadic filter bank, expressed in terms of z-transform polynomials.

Successive Stages of a Dyadic Decomposition

The analysis section of the two-channel filter bank can be replicated successively within the lowpass channel to effect a dyadic decomposition of the Nyquist frequency $[0, \pi]$.

This will generate a succession of intervals

$$(\pi/2^j, \pi/2^{j-1}); j = 1, 2, ..., n,$$

of which the bandwidths are halved repeatedly in the descent from high frequencies to low frequencies. Given T data points, the sequence must end when $T/2^j$ is no longer divisible by 2.

A filter bank with two stages of decomposition is illustrated in Figure 9. Successive stages can be added in a self-evident manner.

If the conditions of sequential and lateral orthogonality prevail within the two-channel network, then they will prevail within and between the channels of the derived network.

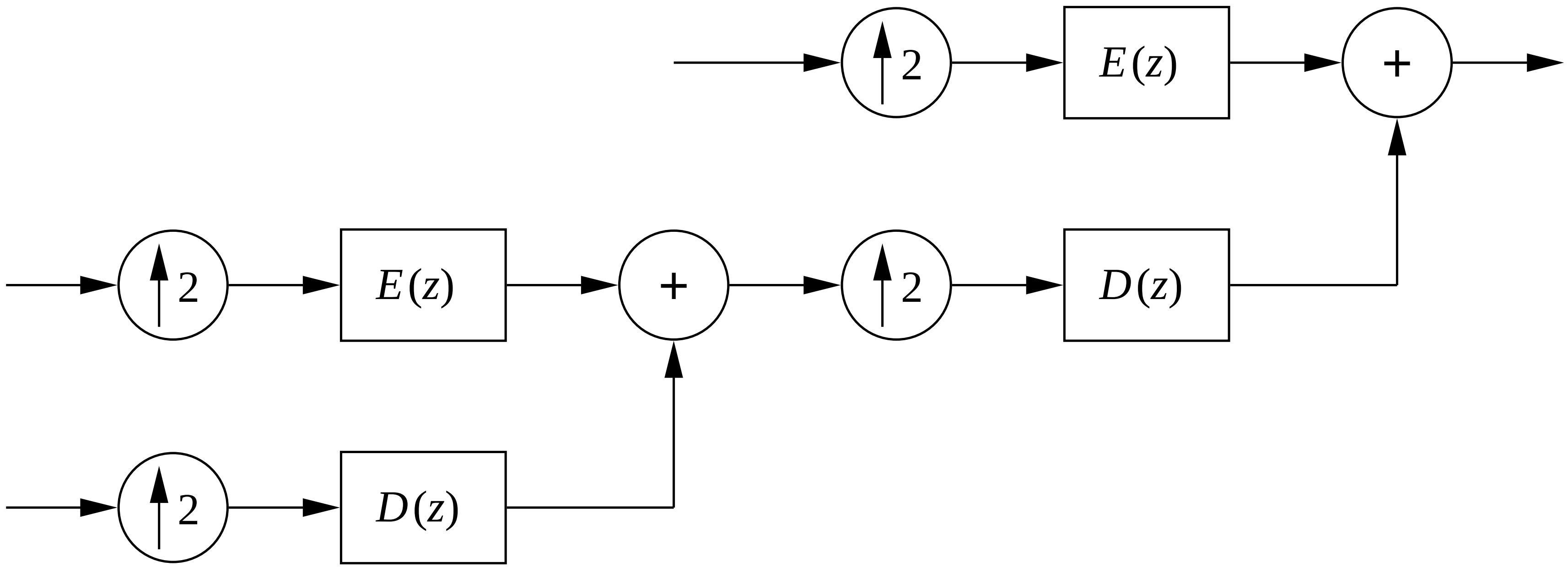

Figure 10. The synthesis section of a dyadic filter bank, expressed in terms of z-transform polynomials.

Reconstruction of a Signal in Successive Stages

The synthesis stage of a dyadic decomposition involves a straightforward reversal of the analysis stage.

The reconstruction of the input signal begins at the lowest frequency level of the decomposition at which a two-channel split has occurred.

By completing the synthesis stage at that level, the split is mended and, if the conditions perfect reconstruction prevail, then the result will be as if the split had not occurred.

Then the split at the next lowest level is mended, and so on upwards, until the input signal has been reconstructed. The synthesis stage to accompany the analysis stage of Figure 9 is illustrated in Figure 10.

Dilations and Image Reversals

A dilation by a factor of 2 of a frequency-domain function supported on the interval $[-\pi, \pi]$ will result in the mirror-image reversal of the components supported on the interval $[\pi/2, \pi] \cup [-\pi, -\pi/2] = [\pi - \pi/2, \pi + \pi/2]$. In the process of circular wrapping, the sub-intervals $[\pi/2, \pi]$ and $[-\pi, -\pi/2]$ will be mapped onto the intervals $[-\pi, 0]$ and $[0, \pi]$ respecitvely.

The dilation will cause the component that was supported on the interval $[-\pi/2, \pi/2]$ to be expanded over the entire Nyquist interval of $[-\pi, \pi]$ without any accompanying image reversal. It will give rise to aliasing unless the original function is confined to a one or other of the halfband intervals $[-\pi/2, \pi/2]$ or $[\pi - \pi/2, \pi + \pi/2]$.

The contents of the low-frequency interval $[-\pi/2^n, \pi/2^n]$ can be extracted via a sequence of lowpass filtering and downsampling operations that isolate the contents of a succession of ever-decreasing intervals

$$[-\pi/2, \pi/2], [-\pi/4, \pi/4], \ldots, [-\pi/2^n, \pi/2^n].$$

To isolate the contents of the interval $[\pi(1-2^{-n}), \pi(1+2^{-n})]$ of the highest frequency, a highpass filter is first applied to isolate the contents of $[\pi/2, 3\pi/2]$. This causes an image reversal. Thereafter, a succession of lowpass filters is applied to isolate a narrowing band on the margins of $\pm\pi$.

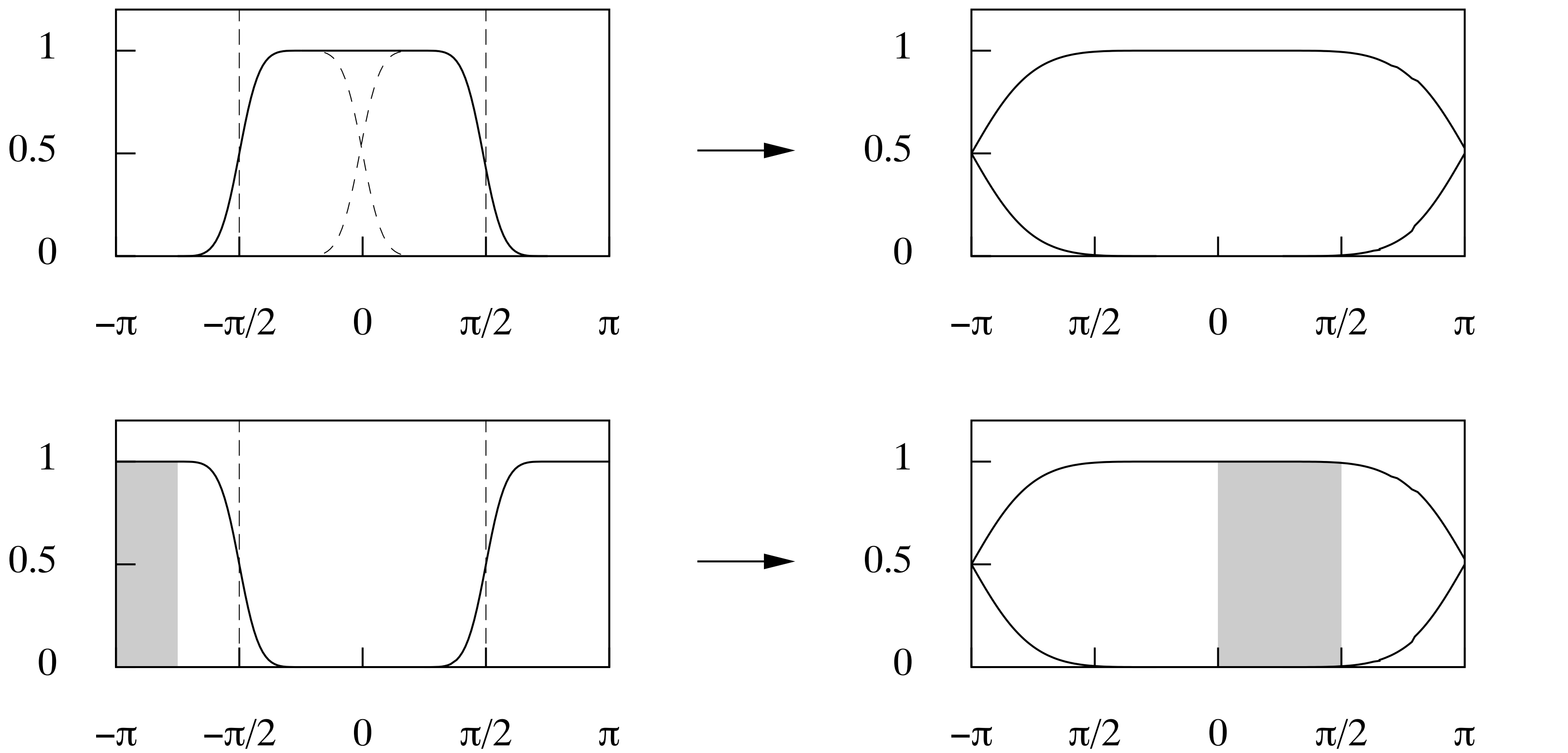

Figure 11. The consequence of dilating the gain functions of the complementary lowpass and highpass filters by a factor of 2 is to create an uniform spectral density function on the interval $[-\pi, \pi]$ of the kind that pertains to a white-noise process.

The Effects of Downsampling on the Periodogram

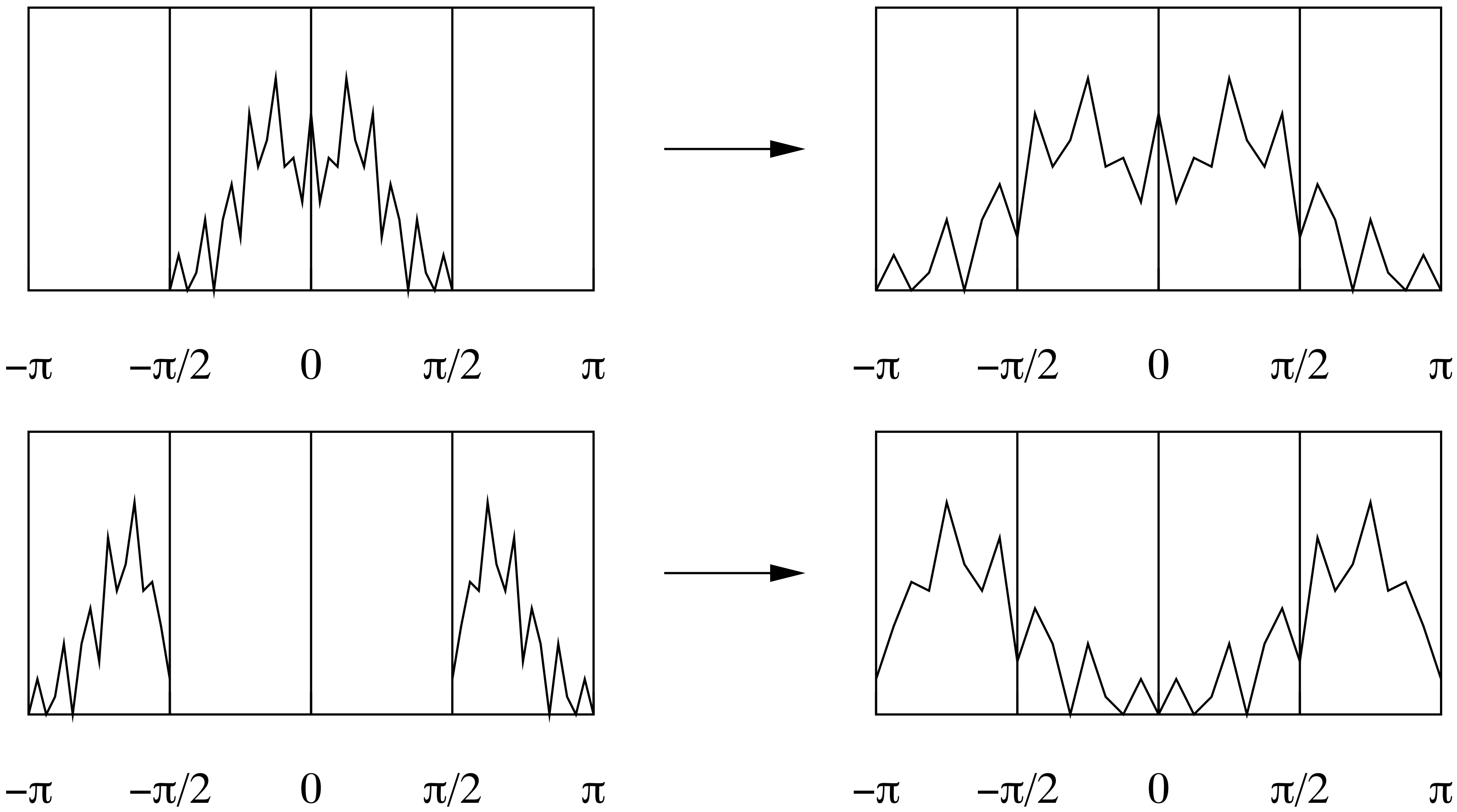

Figure 12. In the process of downsampling, the high frequency content of $[0, \pi]$ will be replaced by a dilated version of its mirror image.

A Decomposition of Equal Bandwidths

A dyadic decomposition that proceeds through n iterations by successively dividing the lowest frequency band generates $n+1$ octave bands.

By splitting all of the frequency bands in n successive iterations, it possible to generate 2^n bands of equal width.

To isolate any of the 2^n bands, except for the band of the lowest frequency, a proper account must be taken of the sequence of image reversals,

Figure 12 will serve to indicate the appropriate sequence of highpass and lowpasss filters.

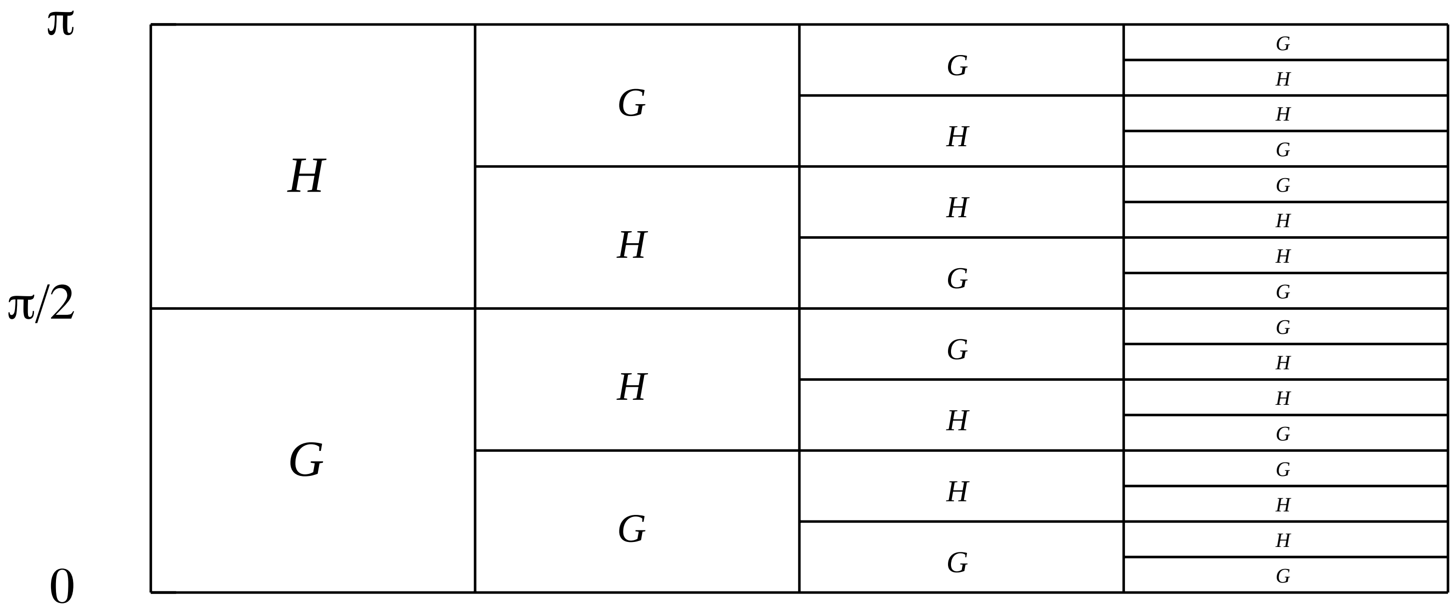

Figure 13. The succession of filters that are employed in dividing the frequency range $[0, \pi]$ into 16 bands of equal width. The lowpass and highpass filters, which are followed by downsampling operations, are indicated by the letters G and H, respectively.

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