BANDPASS FILTERING AND WAVELETS ANALYSIS: TOOLS FOR THE ANALYSIS OF INHOMOGENEOUS TIME SERIES

D. Stephen G. Pollock

University of Leicester

A wavelets analysis provides a means of analysing non-stationary time series of which the underlying statistical structures are continually evolving. It is an analysis both in the time domain and in the frequency domain.

The tutorial will begin by describing the effects of digital filtering in the time domain and the frequency domain. It will proceed to provide the generalisation of the Shannon sampling theorem that is appropriate to bandpass filtering.

This theorem establishes a relationship between continuous signals and their corresponding sampled sequences that is essential to a wavelets analysis. Once this background has been provided, the theories of dyadic and non-dyadic wavelets analysis can be described in detail.

The Frequency Domain

A sequence $y(t) = \{y_t; t = 0, 1, \ldots, T-1\}$ can be projected onto a trigonometrical basis:

$$y_t = \sum_{j=0}^{[T/2]} \rho_j \cos(\omega_j t + \theta_j) = \sum_{j=0}^{[T/2]} \{\alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t)\}.$$

Here, the Fourier frequencies $\omega_j = 2\pi j/T; j = 0, 1, \ldots, [T/2]$ are evenly distributed in the interval $[0, \pi]$ and $[T/2]$ is the integer quotient of the division of T by 2.

Euler's equations enable us to express the trigonometrical functions in terms of complex exponentials:

$$\cos\theta = \frac{e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{-\mathrm{i}}{2}(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}).$$

Define $\zeta_j = (\alpha_j - \beta_j)/2$ and $\zeta_{-j} = (\alpha_j + \beta_j)/2 = \zeta_{T-j}$. Then, we have

$$y_t = \sum_{j=1-[T/1]}^{[T/2]} \zeta_j e^{\mathrm{i}\omega_j t} = \sum_{j=0}^{T-1} \zeta_j e^{\mathrm{i}\omega_j t}.$$

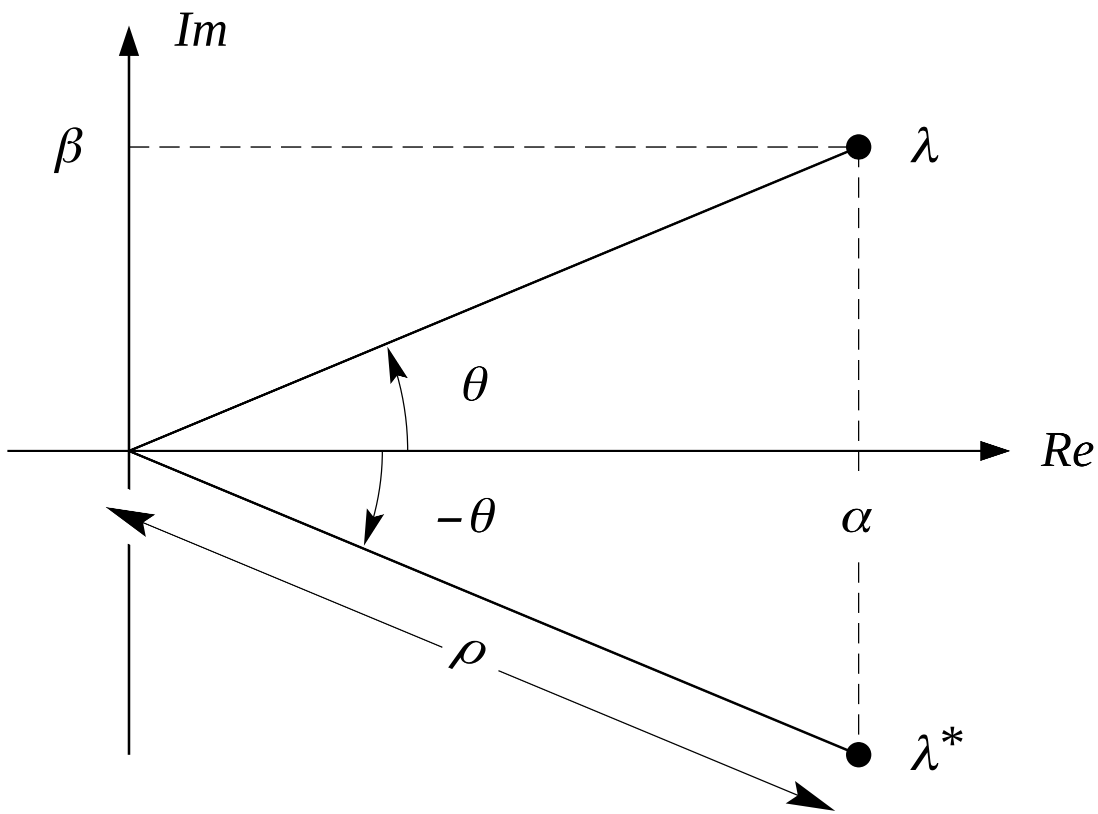


Figure 1. The Argand Diagram showing a complex number $\lambda = \alpha + i\beta$ and its conjugate $\lambda^* = \alpha - i\beta$.

The Periodogram and the Spectrum

The periodogram is the sequence of the squared amplitude coefficients scaled by T. It is also the Fourier transform of the empirical autocovariance function $c(\tau)$:

$$T\rho_j^2 = \sum_{\tau=1-T}^{T-1} c_\tau \cos(\omega_j \tau)$$

A stationary stochastic process also has an expression in terms of trigonometrical functions, described as its spectral representation:

$$y(t) = \int_0^\pi \Big\{ \cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega) \Big\} = \int_{-\infty}^{\infty} e^{i\omega t} dZ(\omega).$$

Here, $dZ(\omega) = \{A(\omega) - B(\omega)\}/2$, where $A(\omega)$, $B(\omega)$ are mutually uncorrelated stochastic processes with infinitesimal increments $dA(\omega)$, $dB(\omega)$ such that

$$E\{dA(\omega)\} = E\{dB(\omega)\} = 0 \quad \text{and} \quad V\{dA(\omega)\} = V\{dB(\omega)\} = 2dF(\omega) = 2f(\omega)d\omega.$$

The analytic function $f(\omega)$ is the spectrum of the process, which is also the Fourier transform of the theoretical autocovariance sequence $\gamma(\tau)$:

$$f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma_\tau \cos(\omega_\tau) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{i\omega_\tau}.$$

A Requirement for Detrending

In a Fourier analysis, the data are treated as a single cycle of a periodic function defined on the real line or on the circumference of a circle. A trended sequence will give rise to a sawtooth function, which has a one-over-f periodogram, with a dominant low-frequency component.

To assess the cyclical elements of the data, one must detrend the data. The ordinates of a linear trend function are given by

$$\begin{aligned} x &= y - Q(Q'Q)^{-1}Q'y \\ &= y - e, \end{aligned}$$

where e is the vector of the residual sequence, and where

$$Q' = \begin{bmatrix} 1 & -2 & 1 & \ldots & 0 & 0 \\ 0 & 1 & -2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & -2 & 1 \end{bmatrix}$$

is the matrix version of the twofold difference operator.

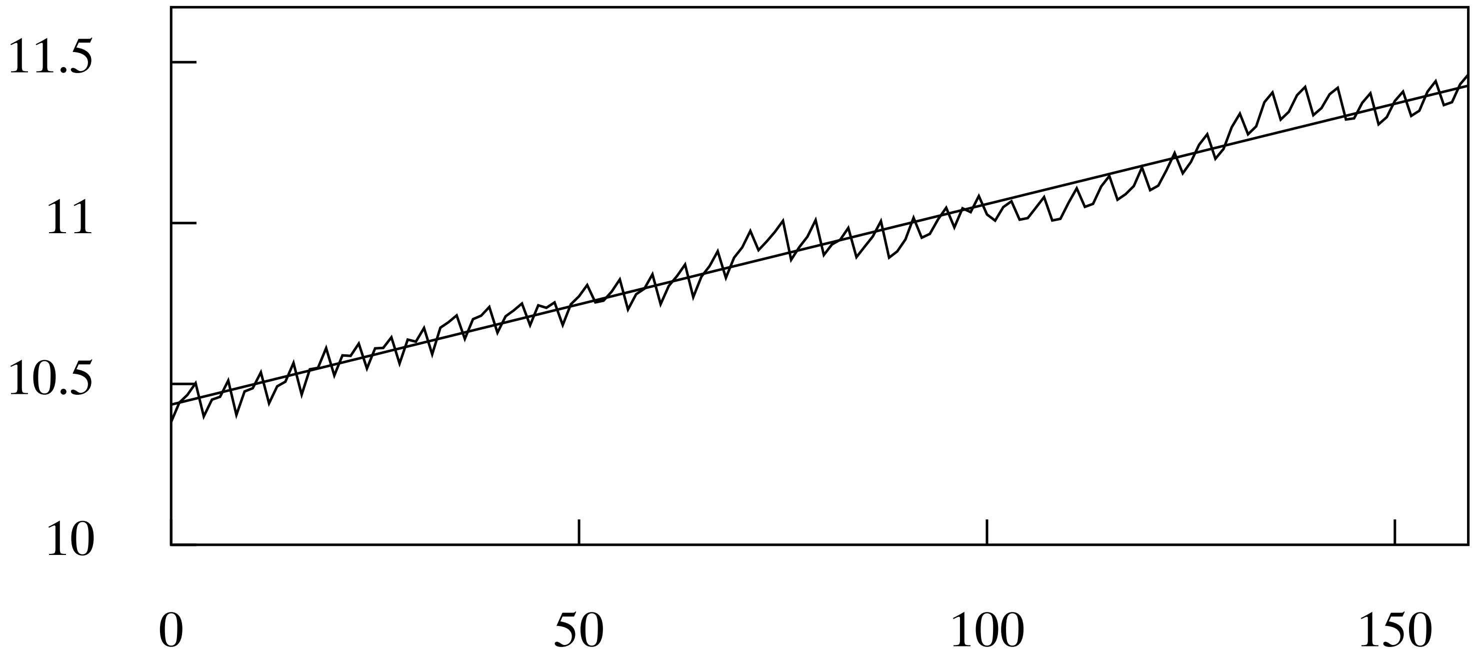


Figure 2. The quarterly sequence of the logarithms of household consumption expenditure in the U.K. for the years 1955 to 1994 with an interpolated linear trend.

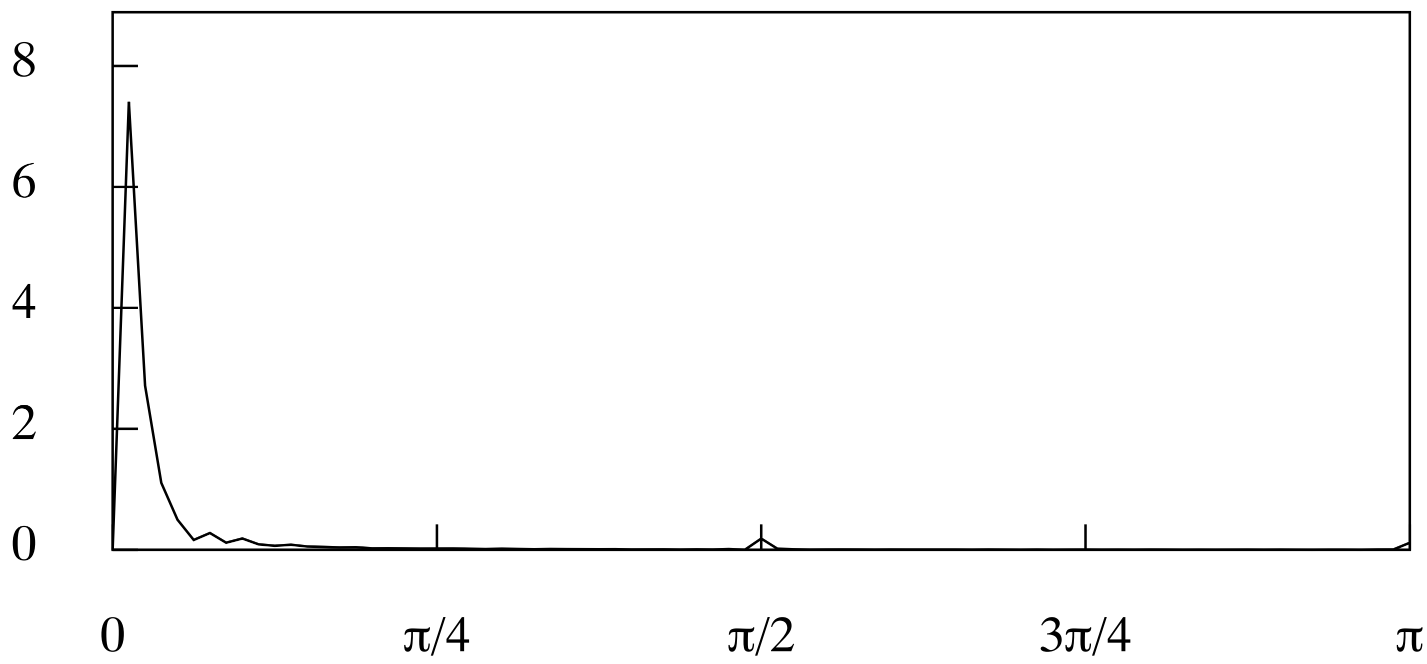


Figure 3. The periodogram of the logarithmic consumption data.

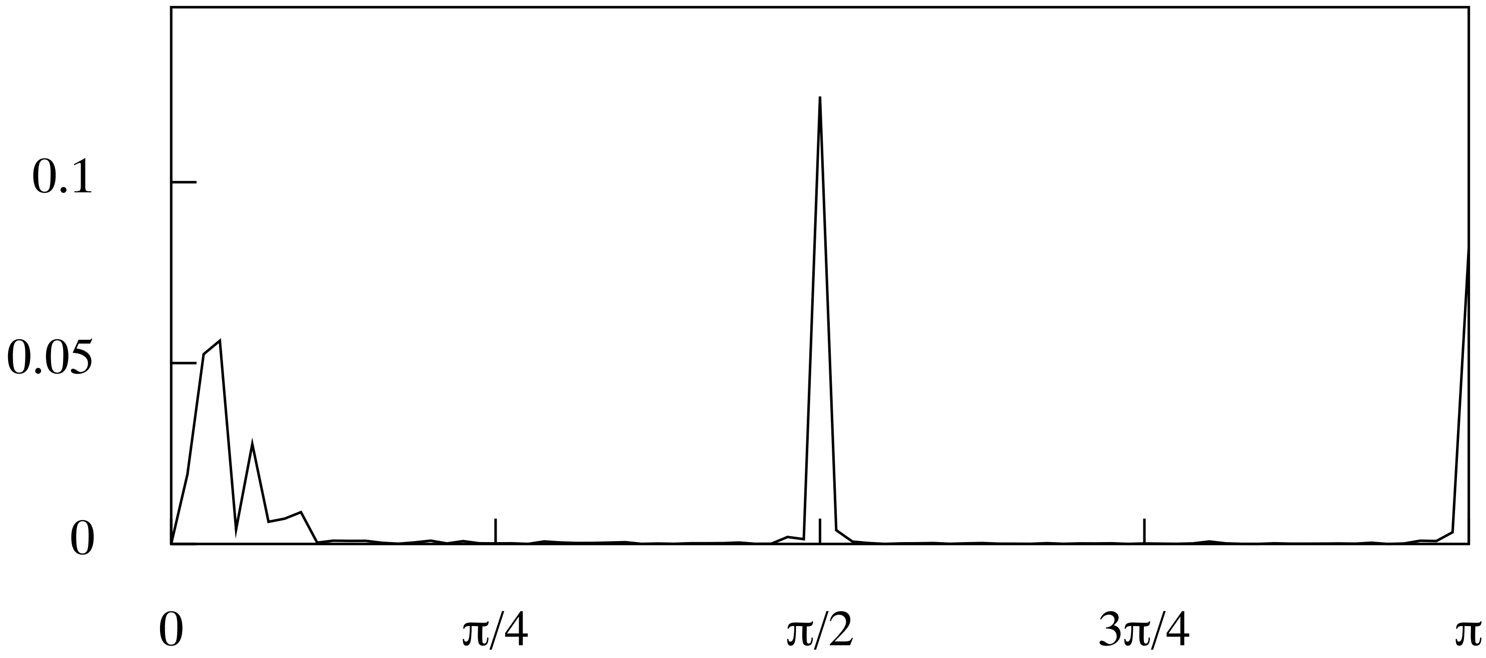


Figure 4. The periodogram of the residual sequence from the linear detrending of the logarithmic consumption data.

z-Transforms and Linear Filters

If a sequence $x(t)$ is absolutely summable, then it has a convergent z-transform:

$$x(z) = \sum_t x_t z^t \quad \text{with} \quad z = e^{-i\omega}.$$

The linear filtering of $x(t)$ entails successive linear combinations of it elements:

$$y(t) = \sum_j \psi_j x(t-j).$$

By associating z^t to each element of y_t and by summing the sequence, we get

$$y_t z^t = \sum_t \left\{ \sum_j \psi_j x_{t-j} \right\} z^t \quad \text{or} \quad y(z) = \psi(z)x(z),$$

where $\psi(z) = \sum_j \psi_j z^t$. The sequence $\{\psi_j\}$ of the filter's coefficients constitutes its response on the output side to the input of a unit impulse. An infinite impulse response (IIR) will commonly entail a recursive equation

$$\sum_{j=0}^{p} \phi_j y_{t-j} = \sum_{j=0}^{q} \theta_j x_{t-j} \quad \text{with} \quad \phi_0 = 1,$$

of which the z-transform is

$$\phi(z)y(z) = \theta(z)x(z).$$

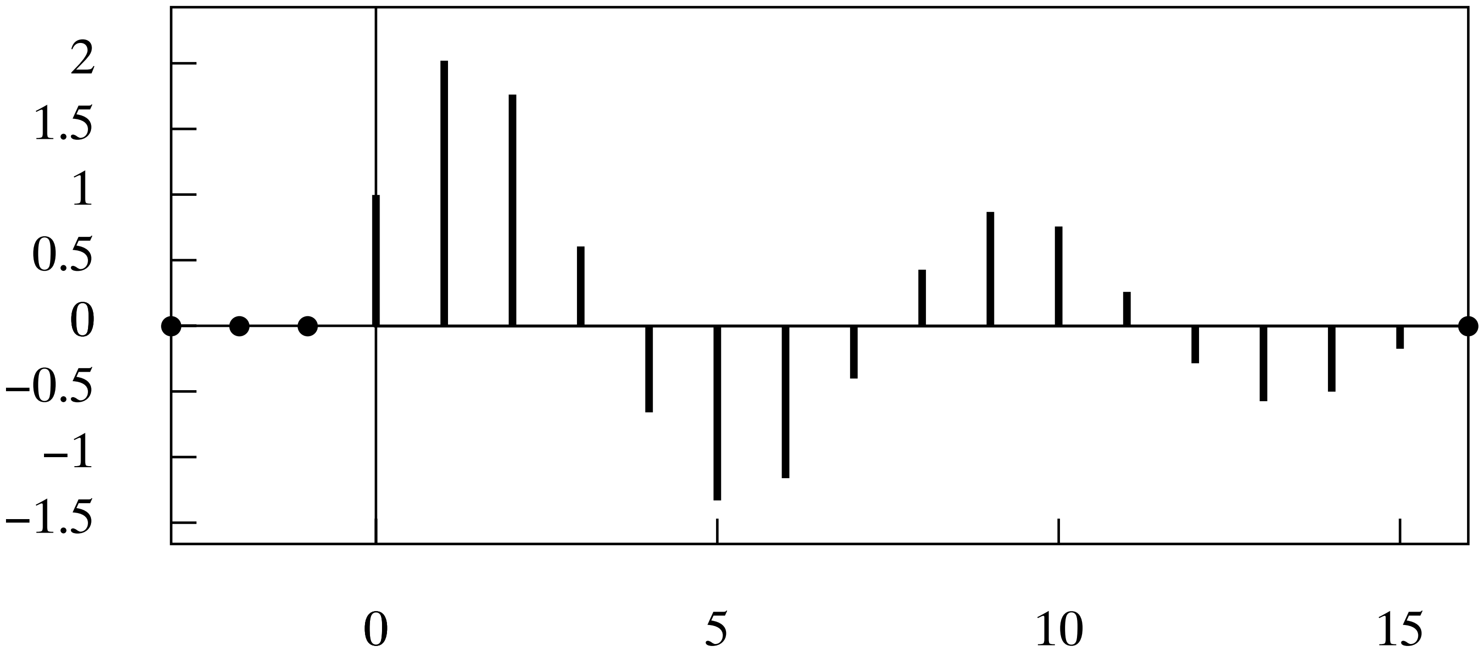


Figure 5. The impulse response of the transfer function $\theta(z)/\phi(z)$ with $\phi(z) = 1.0 - 1.2728z + 0.81z^2$ and $\theta(z)(z) = 1.0 + 0.75z$.

The Response of a Filter to a Sinusoidal Input

Consider mapping the perpetual signal sequence $\{x_t = \cos(\omega t)\}$ through the transfer function with the coefficients $\{\psi_0, \psi_1, \ldots\}$. The output is

$$y(t) = \sum_j \psi_j \cos\big(\omega[t - j]\big).$$

By virtue of the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$, this becomes

$$\begin{aligned} y(t) &= \Big\{\sum_j \psi_j \cos(\omega j)\Big\} \cos(\omega t) + \Big\{\sum_j \psi_j \sin(\omega j)\Big\} \sin(\omega t) \\ &= \alpha \cos(\omega t) + \beta \sin(\omega t) = \rho \cos(\omega t - \theta). \end{aligned}$$

The transfer function has a twofold effect upon the signal. There is a *gain effect,* whereby the amplitude of the sinusoid is increased or diminished by the factor ρ.

There is a *phase effect,* whereby the peak of the sinusoid is displaced by a time delay of θ/ω periods, where $\theta = \tan^{-1} \beta/\alpha$.

The frequency of the output is the same as the frequency of the input, which is a fundamental feature of all linear dynamic systems.

The frequency response of the filter is defined for all values of $\omega \in [-\pi, \pi]$.

The Response of a Filter to a Complex-Exponential Input

The filter's response to a complex exponential input $\exp\{\mathrm{i}\omega t\}$ is represented by

$$y(t) = \sum_j \psi_j e^{\mathrm{i}\omega(t-j)} = \psi(\omega)e^{\mathrm{i}\omega t}.$$

The complex-valued function $\psi(\omega)$, which is the frequency response of the filter, can be expressed as

$$\psi(\omega) = |\psi(\omega)|e^{-\mathrm{i}\theta(\omega)},$$

where $|\psi(\omega)|$ represents the gain effect and $\theta(\omega)$ represents the phase effect. The squared gain is evaluated via

$$\begin{aligned}|\psi(\omega)|^2 &= \Big(\sum_j \psi_j e^{-i\omega j}\Big)\Big(\sum_k \psi_k e^{i\omega k}\Big)\\ &= \sum_\tau \Big(\sum_j \psi_j\psi_{j-\tau}\Big)e^{-i\omega\tau}; \quad \tau = j-k.\end{aligned}$$

This is liable to be written as

$$|\psi(z)|^2 = \psi(z)\psi(z^{-1}) \quad \text{with} \quad z = e^{\mathrm{i}\omega}.$$

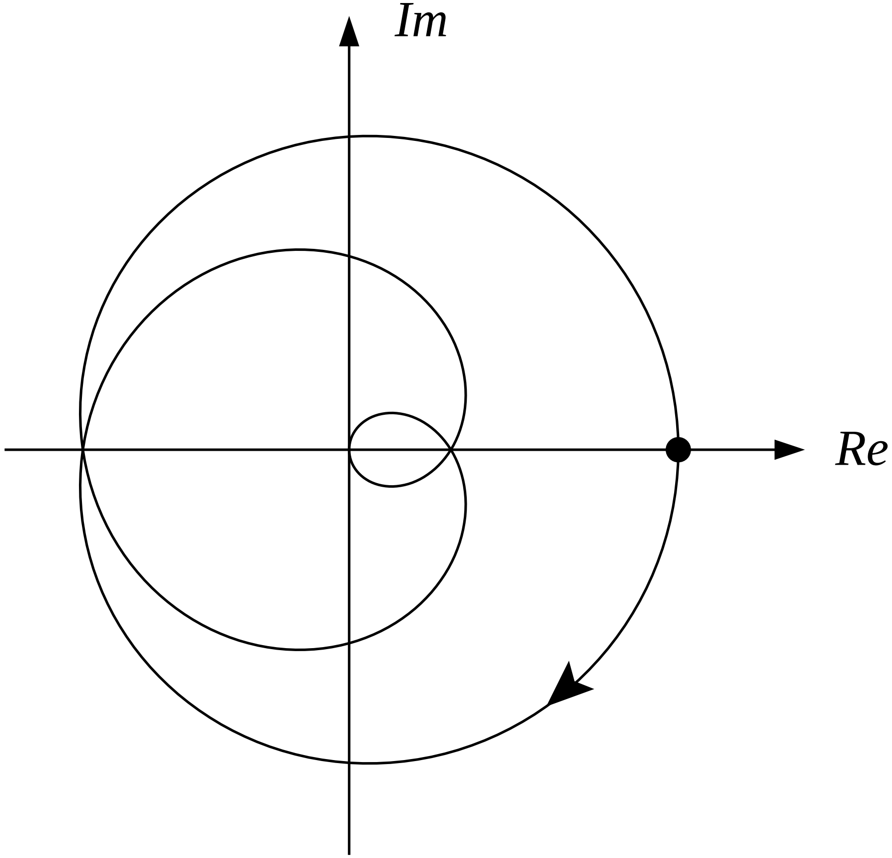


Figure 6. The path described in the complex plane by the frequency-response function $\psi(\omega)$ of the differencing filter $\psi(L) = L^2 - L^3$ as ω increases from $-\pi$ to π. The trajectory originates in the point on the real axis marked by a dot. It traverses the origin when $\omega = 0$ and it returns to the dot when $\omega = \pi$.

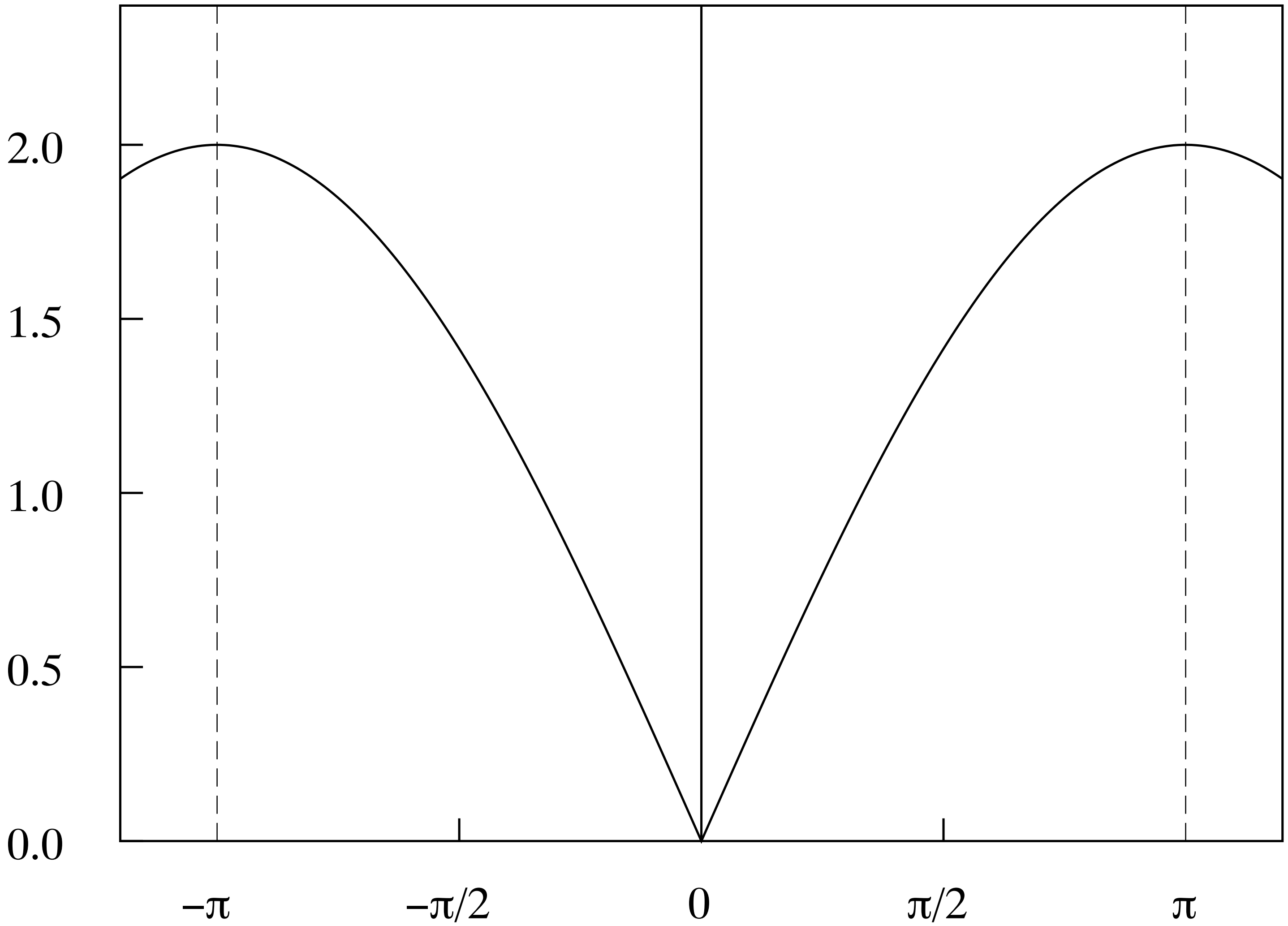


Figure 7. The gain $|\psi(\omega)|$ of the differencing filter $\psi(L) = L^2 - L^3$.

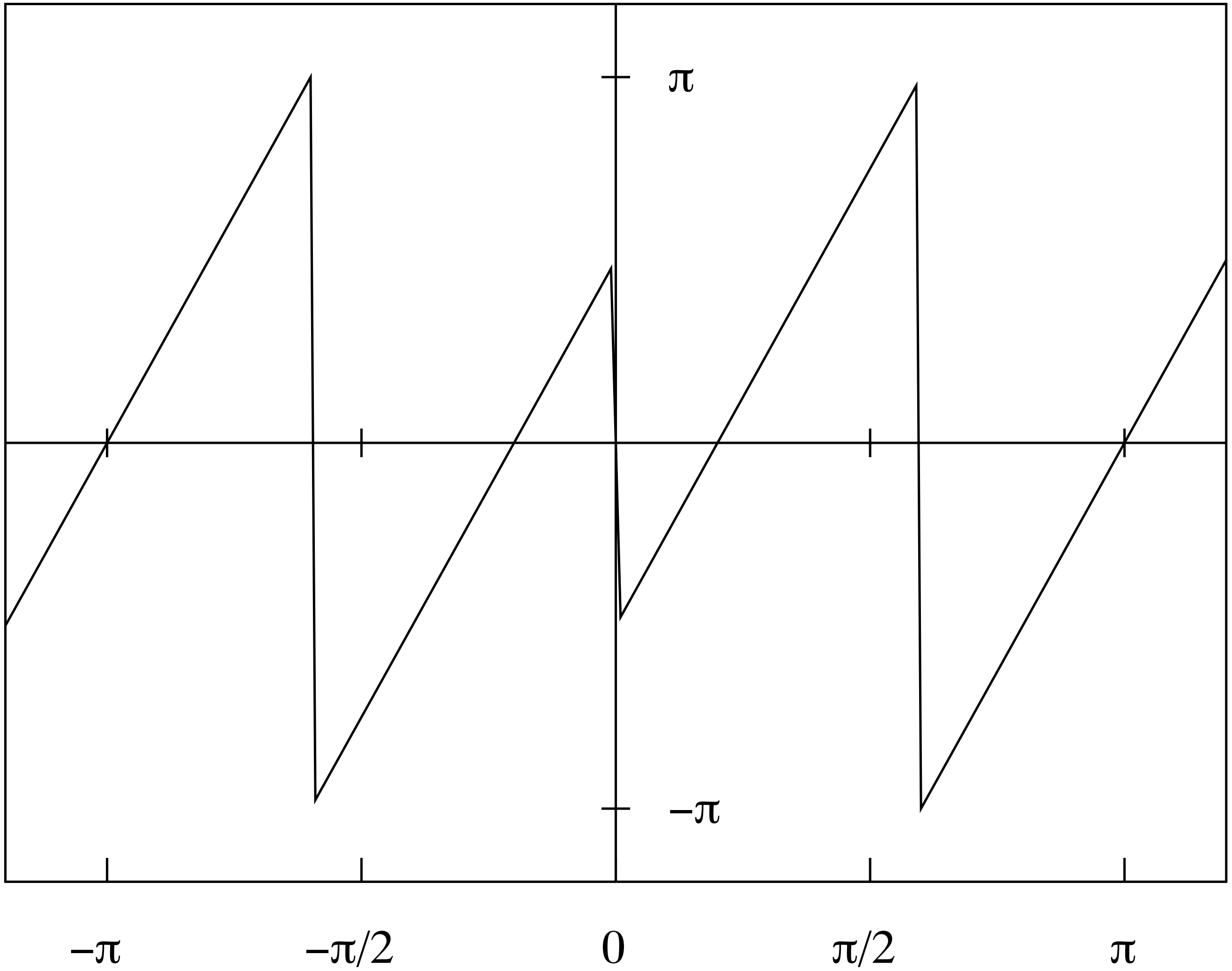


Figure 8. The phase $-\text{Arg}\{\psi(\omega)\}$ of the differencing filter $\psi(L) = L^2 - L^3$.

Frequency Shifting by Cosine Modulation

Consider a function defined as follows in the frequency domain:

$$\psi(\omega) = \begin{cases} 1, & \text{if } |\omega| \in [\alpha, \beta]; \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding time-domain function is

$$\begin{aligned} \psi(t) = \frac{1}{2\pi}\int_{-\beta}^{-\alpha} e^{\mathrm{i}\omega t} d\omega + \frac{1}{2\pi}\int_{\alpha}^{\beta} e^{\mathrm{i}\omega t} d\omega &= \left(\frac{e^{\mathrm{i}\beta t} - e^{-\mathrm{i}\beta t}}{2\pi \mathrm{i}t}\right) - \left(\frac{e^{\mathrm{i}\alpha t} - e^{\mathrm{i}\alpha t}}{2\pi \mathrm{i}t}\right) \\ &= \frac{1}{\pi t}\{\sin(\beta t) - \sin(\alpha t)\}. \end{aligned}$$

The identity $\sin(A + B) - \sin(A - B) = 2\cos A \sin B$ can be used, with $\gamma = (\alpha + \beta)/2$ and $\delta = (\beta - \alpha)/2$, in place of A and B, to show that

$$\begin{aligned} \psi(t) = \frac{1}{\pi t}\{\sin(\beta t) - \sin(\alpha t)\} &= \frac{2}{\pi t}\cos\{(\alpha + \beta)t/2\}\sin\{(\beta - \alpha)t/2\} \\ &= \frac{2}{\pi t}\cos(\gamma t)\sin(\delta t). \end{aligned}$$

The centre of the pass band is γ, whereas δ is half its width. The cut-off points of the pass band are $\alpha, \beta \in [0, \pi]$. The lowpass prototype filter is $2\sin(\delta t)/(\pi t)$ whereas $\cos(\gamma t)$ effects the frequency shifting.

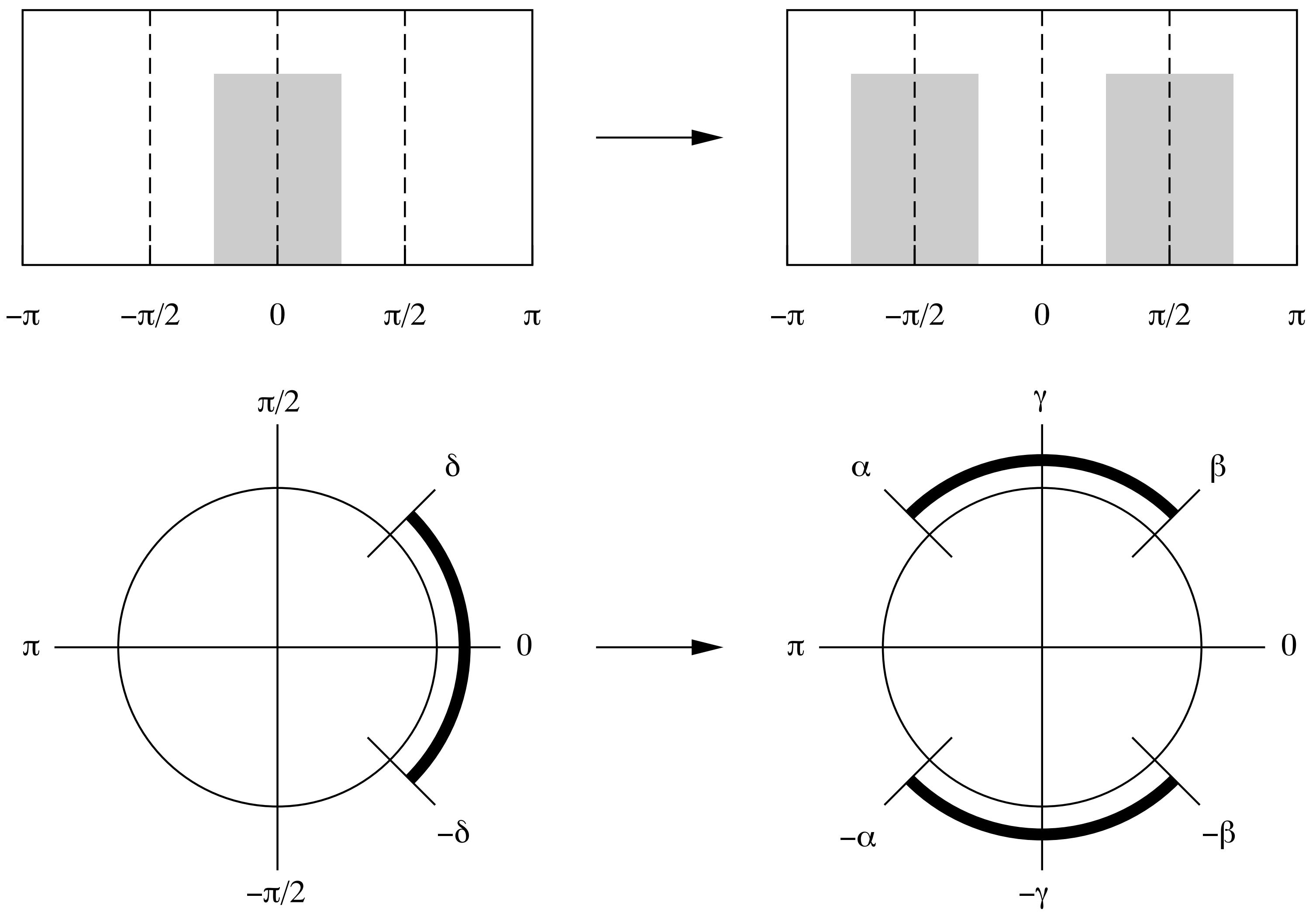


Figure 9. The conversion of a lowpass filter to a bandpass filter by a cosine modulatiom, which creates two copies of the lowpass filter.

The Shannon–Nyquist Sampling Theorem

Let $x(t)$ be a continuous function of time and let $\xi(\omega)$ be its Fourier transform, which is assumed to be band-limited and supported on the interval $[-\pi, \pi]$. Then, $\xi(\omega)$, which may be regarded as a periodic function, has a series expansion

$$\xi(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}\omega k} \quad \text{with} \quad x_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega) e^{\mathrm{i}\omega k} d\omega,$$

where $\{x_k; k = 0, \pm 1, \pm 2, \ldots\}$ are sampled from $x(t)$ at unit intervals. Substituting the LHS into the RHS with the index k therein replaced by the continuous variable t gives

$$\begin{aligned} x(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\{ \sum_{k=-\infty}^{\infty} x_k e^{-\mathrm{i}\omega k} \right\} e^{\mathrm{i}\omega t} d\omega &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} x_k \left\{ \int_{-\pi}^{\pi} e^{\mathrm{i}\omega (t-k)} d\omega \right\} \\ &= \frac{1}{\pi} \sum_{k=-\infty}^{\infty} x_k \frac{\sin\{\pi(t-k)\}}{t-k}. \end{aligned}$$

The final expression shows how the continuous function $x(t)$ can be recovered from its sampled ordinates $\{x_k; k = 0, \pm 1, \pm 2, \ldots\}$.

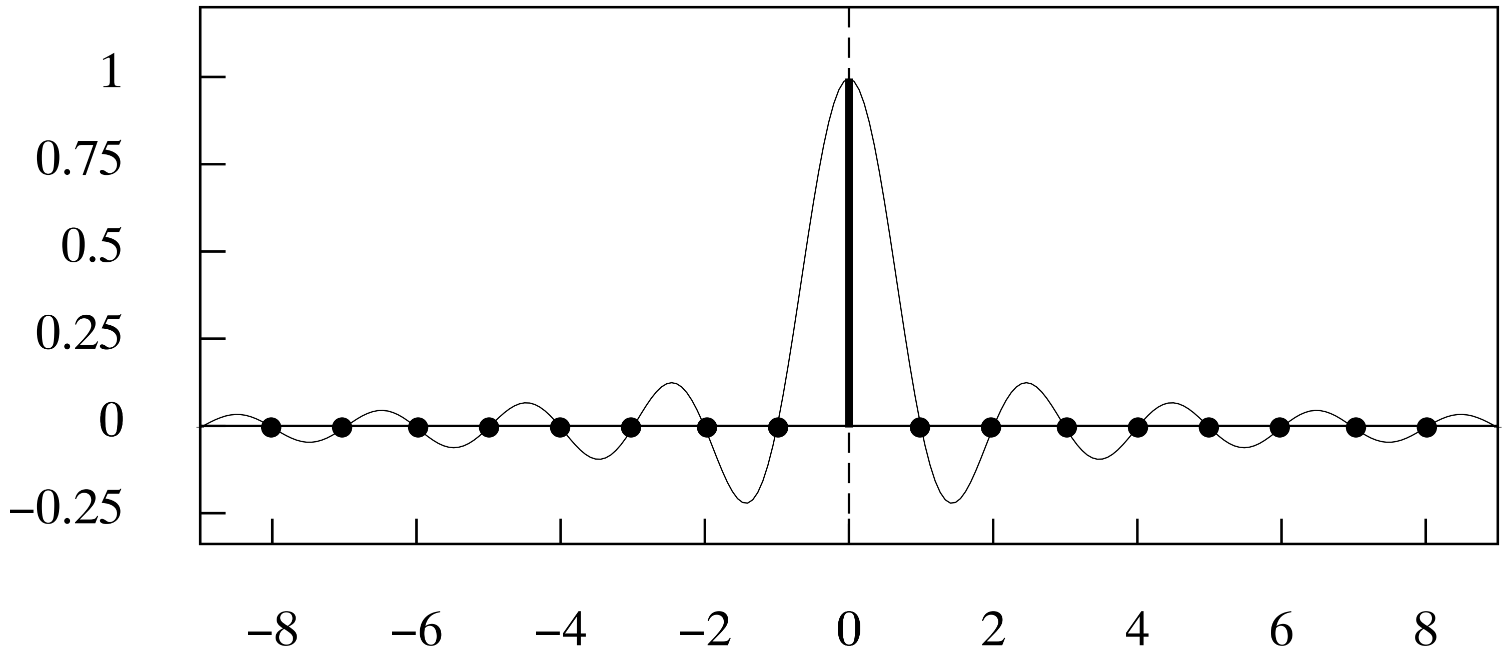


Figure 10. The sinc function $\psi(t) = \sin(\pi t)/\pi t$.

Aliasing and the Shannon–Nyquist Sampling Theorem

In sampling a continuous function, signal frequencies in excess of π radians per sampling interval are confounded with frequencies within the interval $[0, \pi]$.

Consider a cosine wave of a frequency ω in the interval $\pi < \omega < 2\pi$ that is sampled at unit intervals. Let $\omega^* = 2\pi - \omega$. Then,

$$\begin{aligned}\cos(\omega t) &= \cos\left\{(2\pi - \omega^*)t\right\}\\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t)\\ &= \cos(\omega^* t);\end{aligned}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

Let $\xi(\omega)$ be the transform of the continuous aperiodic function $x(t)$, and let $\xi_S(\omega)$ be the transform of the sampled sequence $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$. Then

$$x_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi(\omega) e^{\mathrm{i}\omega t} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi_S(\omega) e^{\mathrm{i}\omega t} d\omega.$$

The equality of the two integrals implies that

$$\xi_S(\omega) = \sum_{j=-\infty}^{\infty} \xi(\omega + 2j\pi).$$

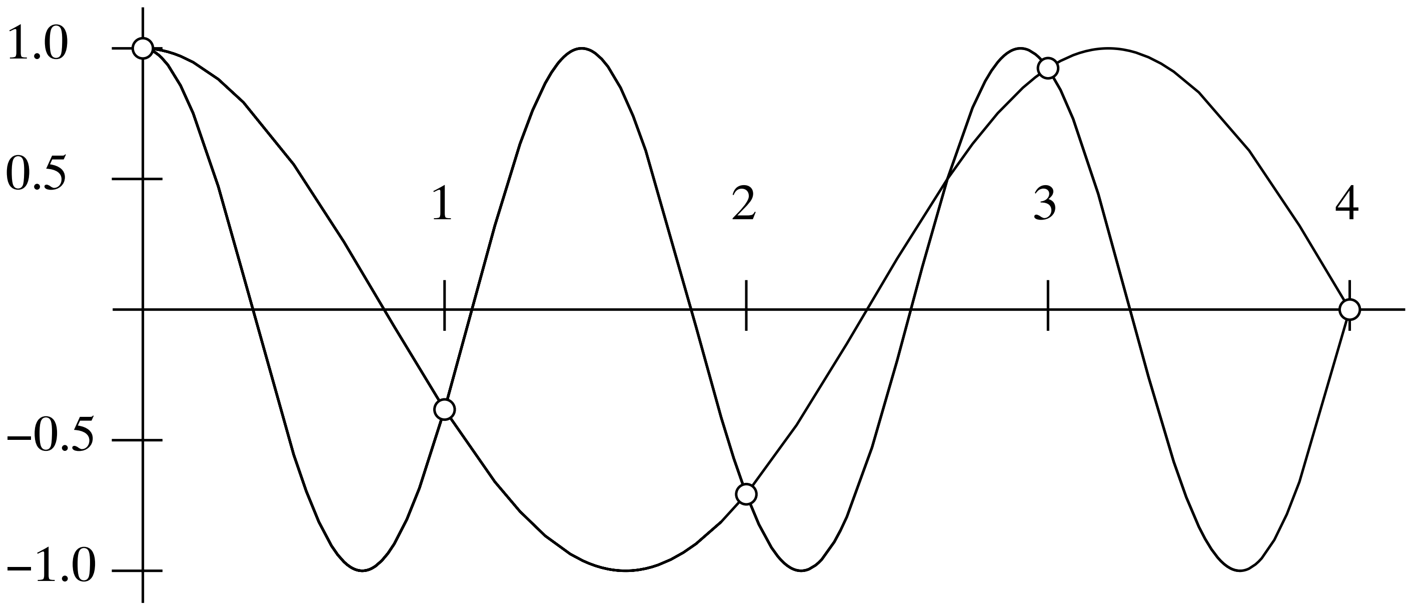


Figure 11. The values of the function $\cos\{(11/8)\pi t\}$ coincide with those of its alias $\cos\{(5/8)\pi t\}$ at the integer points $\{t = 0, \pm 1, \pm 2, \ldots\}$.

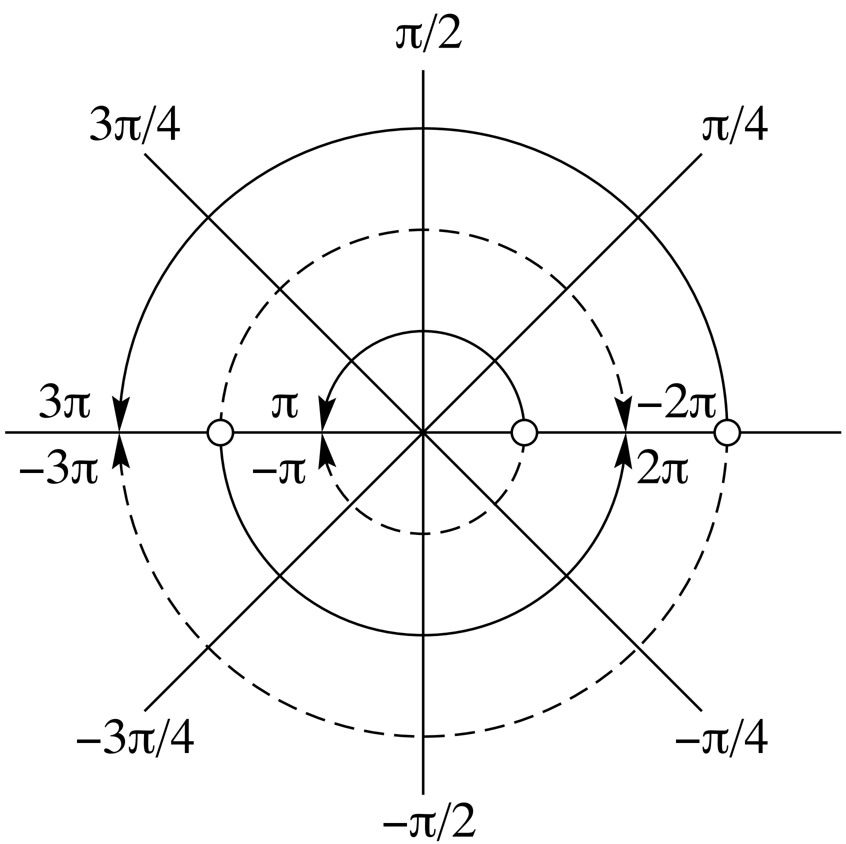


Figure 12. A diagram to illustrate the aliasing of frequencies when the Nyquist frequency is at π radians per sample interval. The arcs with the broken lines correspond to negative frequencies.

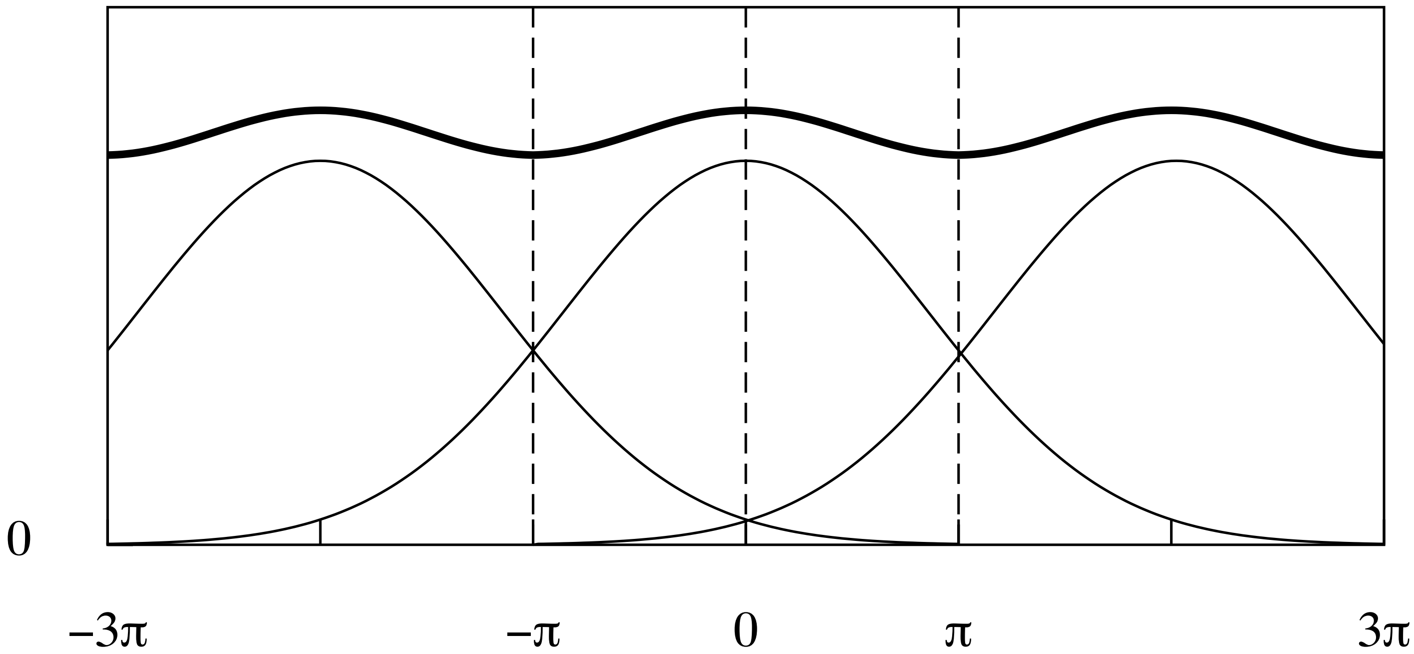


Figure 13. The figure illustrates the aliasing effect of regular sampling. The bell-shaped function supported on the interval $[-3\pi, 3\pi]$ is the spectrum of a continuous-time process. The spectrum of the sampled process, represented by the heavy line, is a periodic function of period 2π.

Upsampling and Downsampling

Given that

$$y(z) = y_0 + y_1 z + y_2 z^2 + y_3 z^3 + y_4 z^4 + \cdots,$$
$$y(-z) = y_0 - y_1 z + y_2 z^2 - y_3 z^3 + y_4 z^4 - \cdots,$$

the z-transforms of the sequence of even-valued elements is obtained from

$$y^e(z^2) = \frac{1}{2}\{y(z) + y(-z)\} = y_0 + y_2 z^2 + y_4 z^4 + y_6 z^6 + \cdots.$$

Replacing the argument z^2 by z gives the downsampled sequence

$$y(\downarrow 2)(z) = \frac{1}{2}\{y(z^{1/2}) + y(-z^{1/2})\} = y_0 + y_2 z + y_4 z^2 + y_6 z^3 + \cdots.$$

The upsampling of a sequence is a matter of interpolating zeros between the elements. Such zeros will not be visible within the z-transform, wherein they would be associated with the odd-valued powers of z.

The elements of the original sequence will now be associated with even-valued powers. Thus, the only effect of the upsampling will be to replace the argument z by z^2 to produce the following *alternant* series:

$$y\{(\downarrow 2) \uparrow 2\}(z) = y_0 + y_2 z^2 + y_4 z^4 + y_6 z^6 + \cdots.$$

The Spectral Effect of the Upsampling and Downsampling

In downsampling, the angular velocity ω is replaced by $\omega/2$. The function is wrapped twice around the circumference of the circle, and the overlying ordinates are added. The effect is one of spectral aliasing.

In upsampling, the frequency argument is multiplied by 2 and the frequency function evolves at twice the rate. The function undergoes two cycles as ω traverses an interval of 2π radians, and two images of the spectrum are mapped into the interval.

The effect of downsampling is summarised by writing

$$p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(\omega/2) + p(\pi + \omega/2)\}.$$

Since $\exp\{\pm i\pi\} = -1$ and $\exp\{-i(\pi + \omega/2)\} = -\exp\{-i\omega/2\}$, this can be expressed, in terms of $z = \exp\{-i\omega\}$, as

$$p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(z^{1/2}) + p(-z^{1/2})\}.$$

The effect of the subsequent upsampling, which doubles the value of the frequency argument, is summarised by $p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(\omega) + p(\pi + \omega)\}$, which can also be written as

$$p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(z) + p(-z)\}.$$

Hysenberg's Principle

Let $\psi(t) \longleftrightarrow \psi(\omega)$ represent a function in the time domain and its Fourier transform in the frequency domain. Let $\psi(t)$ and $\psi(\omega)$ be centred on zero, and with unit integrals in t and in $f = \omega/2\pi$. Then, their dispersions are:

$$\sigma_t^2 = \int t^2 |\psi(t)|^2 dt \quad \text{and} \quad \sigma_\omega^2 = \int \omega^2 |\psi(\omega)|^2 \frac{d\omega}{2\pi}.$$

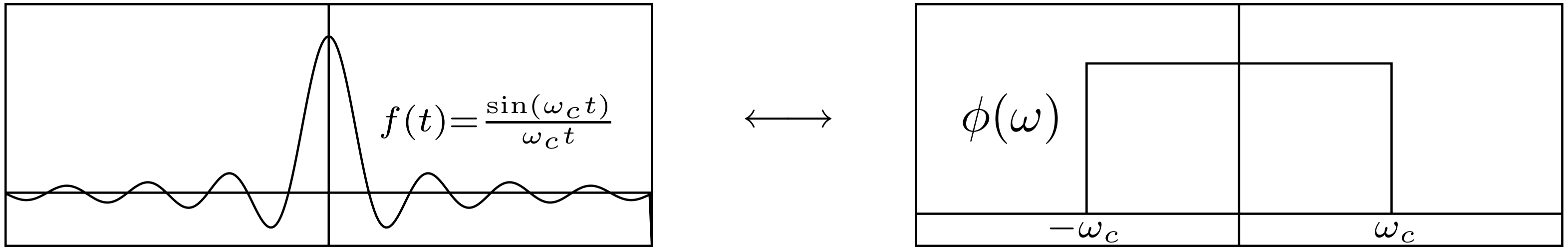
If $\psi(t)$ vanishes faster than $1/\sqrt{t}$, then there is

$$\sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}.$$

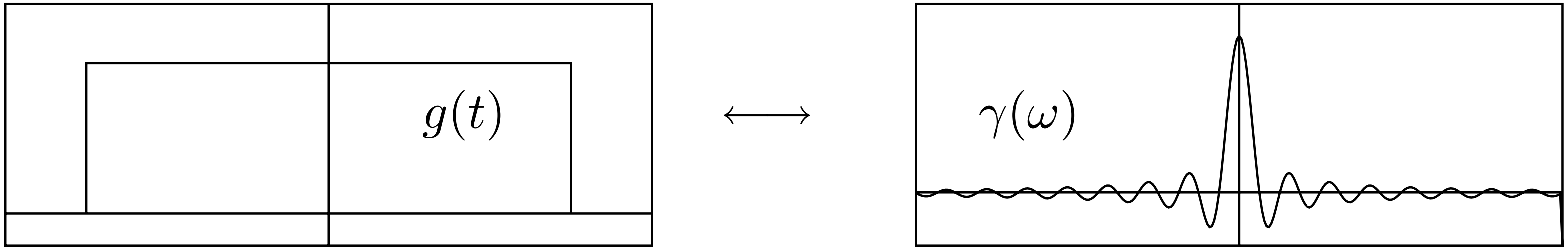
The inequality is commonly described as Heisenberg's uncertainty principle. The equality is attained by a Gaussian signal

$$\psi(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2},$$

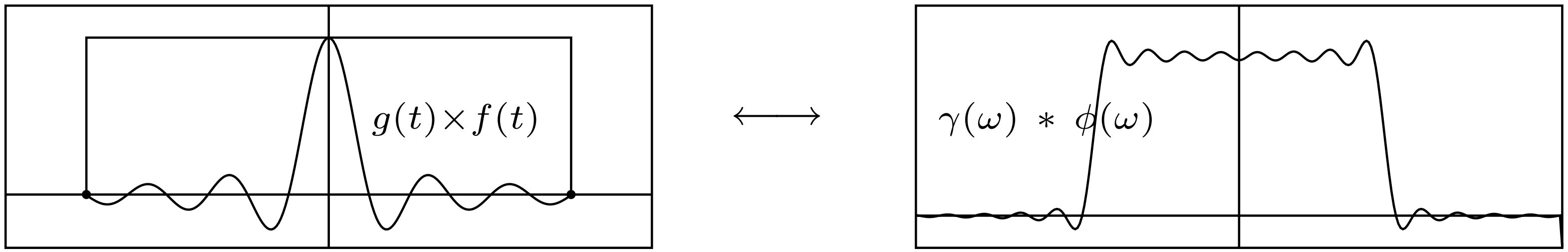
of which the Fourier transform, described in statistics as the characteristic function, also has a Gaussian form.



The Fourier Transform of a time-domain sinc function is a rectangle in the frequency domain.



The Fourier Transform of a time-domain rectangle is a sinc function in the frequency domain.



Truncating the time domain sinc function causes leakage in the frequency domain. The time-domain truncation corresponds to a convolution in the frequency domain.