Filters and Wavelets for Dyadic Analysis

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This chapter describes a variety of wavelets and scaling functions and the manner in which they may be generated. These continuous-time functions might be regarded as a shadowy accompaniment—and even an inessential one—of a discrete-time wavelet analysis that can be recognised as an application of the techniques of multi rate filtering that are nowadays prevalent in communications engineering.

However, as the Shannon–Nyquist sampling theorem has established, there is a firm correspondence between processes in continuous time that are of bounded in frequency and their equivalent discrete-time representations. It is appropriate, therefore, to seek to uncover the continuous-time processes that might be considered to lie behind a discrete-time wavelets analysis.

Apart from the Haar wavelet and the Shannon or sinc function wavelet, which are polar cases that are diametrically opposed, none of the wavelets that we shall consider have closed-form or analytic expressions. To reveal these functions, one must pursue an iterative or recursive process, based on the two-scale dilation equations, which entail the coefficients of the discrete-time filters that are associated with the two-channel filter bank and with the pyramid algorithm of a dyadic wavelets analysis.

The filters in question are the highpass and lowpass halfband filters that are chosen in fulfilment of the conditions of perfect reproduction and of orthogonality. The iterative procedure bequeaths these essential properties of the filters to the continuous-time wavelets and scaling functions.

It follows that the first step in generating a family of wavelets and scaling functions is to understand how to derive filters that satisfy the above-mentioned conditions. This is the concern of the first section of this chapter. A common stating point is with the autocorrelation generating functions of the lowpass filter and of the complementary highpass filter.

These must satisfy the conditions of sequential orthogonality, individually, and the condition of perfect reproduction, jointly. The lowpass filter is found by factorising the lowpass autocorrelation function, whereas the highpass filter is derived from the lowpass filter in a manner that ensures that, together, the two filters will satisfy the condition of lateral orthogonality.

Orthogonality and Perfect Reproduction

The first concern is to derive filters that will give rise to acceptable versions of the scaling function and the wavelet. This is a matter of finding complementary halfband filters that fulfil the canonical properties of perfect reconstruction and of sequential and lateral orthogonality.

In the case of FIR filters, the conditions of orthogonality will be satisfied only by filters with an even even number of coefficients. Such filters cannot be symmetric about a central coefficient. Therefore, with the exception of the two-point Haar filter, they are bound to induce a non-linear phase effect.

To see the necessity of an even number of coefficients, consider $G(z) = g_0 + g_1z+\cdots+g_{M-1}z^{M-1}$ in the case where M is an odd number, as it must be for central symmetry. Then, if $2n = M-1$, there is $p_{2n} = g_0g_{M-1} \neq 0$, since $g_0, g_{M-1} \neq 0$, by definition. Therefore, the condition of sequential orthogonality is violated.

Nevertheless, it is possible to obtain symmetric canonical filters that have an infinite number of coefficients or that correspond to the circularly wrapped versions of such filters, which are applied to circular data sequences,

In the case of the canonical FIR filter, the basic prescription is the one that was originally derived by Smith and Barnwell (1987). Given an appropriate half band lowpass filter $G(z)$, of $M = 2n$ coefficients, the corresponding highpass filter must fulfil the condition that

$$H(z) = -z^{M-1}G(-z^{-1}) \quad \text{or, equivalently, that} \quad z^{M-1}H(-z^{-1}) = G(z). \tag{1}$$

Thus, the highpass filter is obtained from the lowpass filter by what has been described an alternating flip—namely by the reversal of the sequence of coefficients followed by the application of alternating positive and negative to the elements of the reversed sequence.

The synthesis filters to accompany these analysis filters are just their time-reversed versions:

$$D(z) = G(z^{-1}) \qquad \text{and} \qquad E(z) = H(z^{-1}). \tag{2}$$

The relationship between the polynomials $G(z)$ and $H(z)$ can be illustrated adequately by the case where $M = 4$. The coefficients of the cross covariance function $R(z) = G(z)H(z^{-1})$ can be obtained from the following table:

$$\begin{array}{c|cccc} & g_0 & g_1z & g_2z^2 & g_3z^3 \\ \hline h_0 = g_3 & g_3g_0 & g_3g_1z & g_3g_2z^2 & g_3^2z^3 \\ h_1z^{-1} = -g_2z^{-1} & -g_2g_0z^{-1} & -g_2g_1 & -g_2^2z & -g_2g_3z^2 \\ h_2z^{-2} = g_1z^{-2} & g_1g_0z^{-2} & g_1^2z^{-1} & g_1g_2 & g_1g_3z \\ h_3z^{-3} = -g_0z^{-3} & -g_0^2z^{-3} & -g_0g_1z^{-2} & -g_0g_2z^{-1} & -g_0g_3 \end{array} \tag{3}$$

The coefficients of $R(z)$ associated with the various powers of z are obtained by summing the element in the rows that run in NW–SE direction. It can be seen that $r_0 = 0$, $r_2 = 0$ and $r_{-2} = 0$. More generally, there is

$$R(z) + R(-z) = G(z)H(z^{-1}) + G(-z)H(-z^{-1}) = 0. \tag{4}$$

This the downsampled version of the cross-covariance function, and the equation indicates that the coefficients associated with odd-valued powers of z are zeros.

In the case of symmetric IIR filters , the prescription is that, given an appropriate lowpass filter $G(z) = G(z^{-1})$, the corresponding highpass filter must fulfil the condition that

$$H(z) = -z^{-1}G(-z) \qquad \text{or, equivalently, that} \qquad H(-z) = z^{-1}G(z). \tag{5}$$

Then, the synthesis filters are

$$D(z) = G(z) \qquad \text{and} \qquad E(z) = -zG(-z). \tag{6}$$

There is a manifest similarity in the two cases.

Given these specifications, is easy to see that, if $P(z) = G(z)G(^{-1})$ and $Q(z) = H(z)H(^{-1})$ then, in either case, there is

$$\begin{aligned} P(z) + P(-z) &= G(z)G(z^{-1}) + G(-z)G(-z^{-1}) \\ &= G(z)G(z^{-1}) + H(z)H(z^{-1}) \\ &= H(-z)H(-z^{-1}) + H(-z)H(-z^{-1}) = Q(-z) + Q(z). \end{aligned} \tag{7}$$

Given that $P(z) + P(-z) = 2$, in consequence of the customary the normalisation $p_0 = 1$ of the leading coefficient of $P(z)$, the fist equality denotes the sequential orthogonality of the coefficient of the lowpass filter as displacements that are multiples of two points. The second equality denotes the power complementarity of the two filters. The third equality denotes the sequential orthogonality of the highpass filter.

Successive Approximations to a Wavelet

Except in the polar cases of the Haar wavelet and the Shannon or sinc function wavelet, an explicit functional form for the wavelet or the scaling function is unlikely to be available. Nor is there, in most practical applications, an indispensable requirement to represent of these functions graphically. Nevertheless, it is enlightening to examine the profiles of the wavelets and the scaling functions and to consider a method of calculating them.

Given the filter coefficients, the calculation of the wavelet depends on the basic two–scale dilation equation. This equation expresses the scaling function at one level of resolution in terms of the functions at the twice that resolution:

$$\phi(t) = \sqrt{2}\sum_{k=0}^{M-1} g_k\phi(2t - k). \tag{8}$$

A similar equation is expresses the corresponding wavelet in terms of the scaling functions:

$$g(t) = \sqrt{2}\sum_{k=0}^{M-1} h_k\phi(2t-k). \tag{9}$$

These equations have direct frequency-domain counterparts. Thus, taking the Fourier transform on both sides of (8) gives

$$\begin{aligned}
\phi(\omega) &= \int \phi(t)e^{-\mathrm{i}\omega t}dt = \sqrt{2}\int\sum_{k=0}^{M-1} g_k\phi(2t-k)e^{-\mathrm{i}\omega t}dt \\
&= \sqrt{2}\sum_{k=0}^{M-1}\int \frac{1}{2}g_k\phi(\tau)e^{-\mathrm{i}\omega\tau/2}e^{-\mathrm{i}\omega k/2}d\tau \\
&= \frac{1}{\sqrt{2}}\sum_{k=0}^{M-1} g_k e^{-\mathrm{i}\omega k/2}\int \phi(\tau)e^{-\mathrm{i}\omega\tau/2}d\tau,
\end{aligned} \tag{10}$$

where $\tau = 2t$ had been defined, which has entailed the change of variable technique. This has introduced the factor $dt/d\tau = 1/2$. The term $\exp\{-\mathrm{i}k/2\}$ relates to the half-point displacements in time of the scaling functions. This equation can be summarised using

$$G(\omega) = \sum_j g_j e^{-\mathrm{i}\omega j}, \tag{11}$$

which is the discrete-time Fourier transform of the sequence of the coefficients of the dilation equation. It is also the result of setting $z = \exp\{-\mathrm{i}\omega\}$ in $G(z)$. Also, there is

$$\int_{-\infty}^{\infty} \phi(t)e^{-\mathrm{i}\omega t}dt = \phi(\omega), \tag{12}$$

which is the Fourier transform of $\psi(t)$. With these definitions, equation (19) can be written as

$$\phi(\omega) = \frac{1}{\sqrt{2}}G(\omega/2)\phi(\omega/2), \tag{13}$$

Successive approximations to the wavelet can be generated using the basic time-domain two-scale dilation equation. The iterations are defined by

$$\phi^{(j+1)}(t) = \sqrt{2}\sum_{k=0}^{M-1} g_k\phi^{(j)}(2t-k), \tag{14}$$

were the initial value $\phi^{(0)}$ must be given This may be specified as a unit rectangle. The frequency-domain form of the equation is seen to be

$$\phi^{(j+1)}(\omega) = \frac{1}{\sqrt{2}}G(\omega/2)\phi^{(j)}(\omega/2), \tag{15}$$

The limit of successive iterations or back substitutions, if it exists, is

$$\phi(\omega) = \left[\prod_{j=1}^{\infty} \left\{ \frac{1}{\sqrt{2}} G\left(\frac{\omega}{2^j} \right) \right\} \right] \phi(0). \tag{16}$$

The Haar Wavelets

The Haar function has often been used as a didactic device for introducing the concept of a multi resolution wavelets analysis. It has the twofold advantage that its functional form is readily accessible and that it is easy to use in a multi resolution wavelet analysis.

The Haar wavelet has a finite support in the time domain, but it has the disadvantage that its frequency-domain counterpart, i.e. its Fourier transform, has an infinite support and that it has only a hyperbolic rate of decay.

An opinion that is offered in this text is that, for the purpose of capturing the essentials of a wavelets analysis, one might as well exploit the so-called Shannon wavelets that have characteristics that are the opposites of those of the Haar wavelets.

The Shannon wavelets, which are based on the sinc function, have a finite support in the frequency domain and an infinite support in the time domain. They have a hyperbolic rate of decay in the time domain. They are also analytic functions.

Our description of the Haar wavelet, in this section, will be followed by that of the Shannon wavelets in the next section. In both of these cases, closed form expressions are available for the wavelet and the scaling functions, of which the sampled ordinates are the coefficients of the corresponding filters. Therefore, little attention needs to be given to the lowpass autocorrelation function, which is the usual starting point in the derivation of a system of wavelets.

The Haar scaling function in $\mathcal{V}_0$ is given by

$$\phi(t) = \begin{cases} 1, & \text{if } 0 < t < 1; \\ 0, & \text{otherwise,} \end{cases} \tag{17}$$

and the wavelet in $\mathcal{W}_0$ is given by

$$\psi(t) = \begin{cases} 1, & \text{if } 0 < t < 1/2; \\ -1, & \text{if } 1/2 \leq t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

The scaling function has a unit area $\langle \phi(t), \phi(t) \rangle = 1$. Functions at different integer displacements have no overlap and they are therefore mutually orthogonal with $\langle \phi(t-j), \phi(t-k) \rangle = \delta(j-k)$. Thus, the set of scaling functions functions at integer displacements provides an orthonormal, basis for the space $\mathcal{V}_0$.

For the wavelet there is, likewise, $\langle\psi(t),\psi(t)\rangle = 1$ and $\langle\psi(t-j),\psi(t-k)\rangle = \delta(j-k) = 0$; and the set of wavelet functions at integer displacements provides an orthonormal basis for the space $\mathcal{W}_0$.

The wavelets and the scaling functions are laterally orthogonal, both in alignment, and at integer displacements so that $\langle\psi(t-j),\phi(t-k)\rangle = 0$ for all j,k. The inner product $\langle\phi(t),\psi(t)\rangle = 0$ can be seen as the average of the wavelet over the interval $[0,1]$, which is clearly zero, and the functions at different displacements do not overlap.

The set of basis functions for the nested spaces $\mathcal{V}_j; j := 0,1,\ldots,m$, for which $\mathcal{V}_{j+1} \subset \mathcal{V}_j$, are provided by the dilated scaling functions

$$\phi_{j,k}(t) = 2^{-j/2}\phi(2^{-j}t-k). \tag{19}$$

Here, j is the so-called scale factor, which corresponds to the level of a dyadic decomposition, whereas k is the displacement factor. The actual displacement in the level-j wavelet $\psi_{j,k}(t)$ is by $t = 2^jk$ points, which is its central value that solves the equation $2^{-j}t - k = 0$.

The basis functions for the wavelet spaces $\mathcal{W}_j; j := 0,1,\ldots,m$ are provided by the dilated wavelet function

$$\psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t-k). \tag{20}$$

The first stage of the dyadic decomposition of $\mathcal{V}_0 = \mathcal{V}_1 \oplus \mathcal{W}_1$ produces a direct sum of a space of scaling functions $\mathcal{V}_1$ and a space wavelets $\mathcal{W}_1$, wherein the respective basis functions are

$$\phi_{1,k}(t) = \frac{1}{\sqrt{2}}\phi(t/2-k) \quad \text{and} \quad \psi_{1,k}(t) = \frac{1}{\sqrt{2}}\phi(t/2-k). \tag{21}$$

The two-scale dilation equation for the scaling function $\phi_{1,0}(t)$ is

$$\phi_{1,0}(t) = \frac{1}{\sqrt{2}}\phi(t/2) = g_0\phi(t) + g_1\phi(t-1), \tag{22}$$

and that of the wavelet function $\psi_{1,0}(t)$ is

$$\psi_{1,0}(t) = \frac{1}{\sqrt{2}}\psi(t/2) = h_0\phi(t) + h_1\phi(t-1). \tag{23}$$

Here,

$$g_0 = \frac{1}{\sqrt{2}},\ g_1 = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad h_0 = \frac{1}{\sqrt{2}},\ h_1 = \frac{-1}{\sqrt{2}}. \tag{24}$$

These are the values of the ordinates sampled from the functions $\phi_{1,0}(t)$ and $\psi_{1,0}(t)$ at the points $\epsilon, 1+\epsilon$, where $\epsilon \in (0,1)$. (The inclusion of a small positive displacement ϵ is to avoid taking a sample at the jump point of $\psi_{1,0}(t)$ at $t = 1$.) They are also the values of the filter coefficients of a discrete-time analysis.

It will be observed that

$$g_0^2 + g_1^2 = 1, \quad \text{that} \quad h_0^2 + h_1^2 = 1 \quad \text{and that} \quad g_0 h_0 + g_1 h_1 = 0. \tag{25}$$

which demonstrates that the filters can be used in constructing an orthonormal basis for a discrete-time analysis.

Sinc Function Filters and Shannon Wavelets

The ideal halfband filters and the corresponding Shannon wavelets and scaling functions are realised by virtue of the sinc function. The function provides a prototype for a wavelet that is tractable from the point of view of its mathematical analysis. The wavelet and the scaling function have closed-form analytic expressions that leads directly, via the sampling theorem, to expressions for the filters that are entailed by the two-scale dilation equations.

The sinc function is defined by

$$\begin{aligned}\phi(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}\omega t} d\omega = \frac{1}{2\pi}\left[\frac{e^{\mathrm{i}\omega t}}{\mathrm{i}t}\right]_{-\pi}^{\pi} \\ &= \left(\frac{e^{\mathrm{i}\pi t} - e^{-\mathrm{i}\pi t}}{2\pi \mathrm{i}t}\right) = \frac{\sin(\pi t)}{\pi t}.\end{aligned} \tag{26}$$

This is the (inverse) Fourier transform of a rectangle of unit height supported, in the frequency domain, of the interval $[-\pi, \pi]$. The function, together with its ordinates sampled at unit intervals, is represented in Figure 1.

The sinc function gives rise to a family of scaling functions $\phi(t-k)$ that provide an orthonormal basis for the space $\mathcal{V}_0$. The sequential orthonormality of the functions at unit displacements follows from their fulfilment of the condition that

$$\sum_k |\phi(\omega + 2k\pi|^2 = 1, \qquad \omega \in [-\pi, \pi], \tag{27}$$

which is the frequency-domain equivalent of the condition $\langle \phi(t-j)\phi(t-k)\rangle = \delta(j-k)$, as is indicated under (6.4).

The basis functions for the subspace $\mathcal{V}_1 \subset \mathcal{V}_0$ are given by

$$\phi_{1,k}(t) = 2^{-1/2}\phi(2^{-1}t - k). \tag{28}$$

Substituting the formula of (35) gives

$$\phi_{1,k}(t) = \frac{1}{\sqrt{2}}\left\{\frac{\sin(\pi t/2 - k)}{\pi t/2}\right\} = \sqrt{2}\frac{\sin(\pi t/2 - k)}{\pi t}. \tag{29}$$

The scaling function $\phi_{1,0}(t)$, together with its ordinates sampled at unit intervals, is represented in Figure 2.

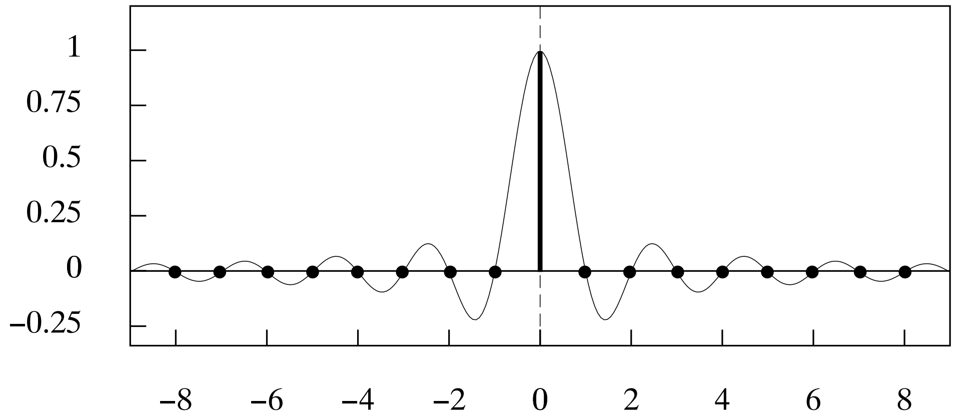

Figure 1. The scaling function $\phi_{(0)}(t)$.

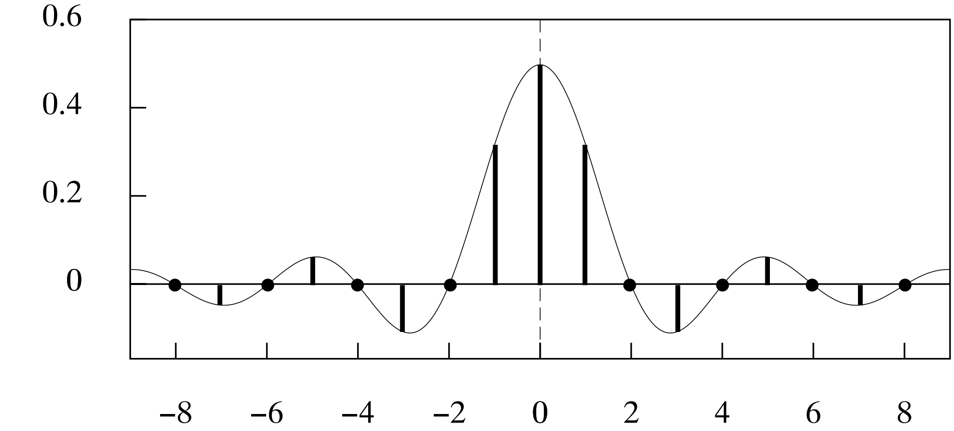

Figure 2. The scaling function $\phi_{(1)}(t) = \phi_{(0)}(t/2)$.

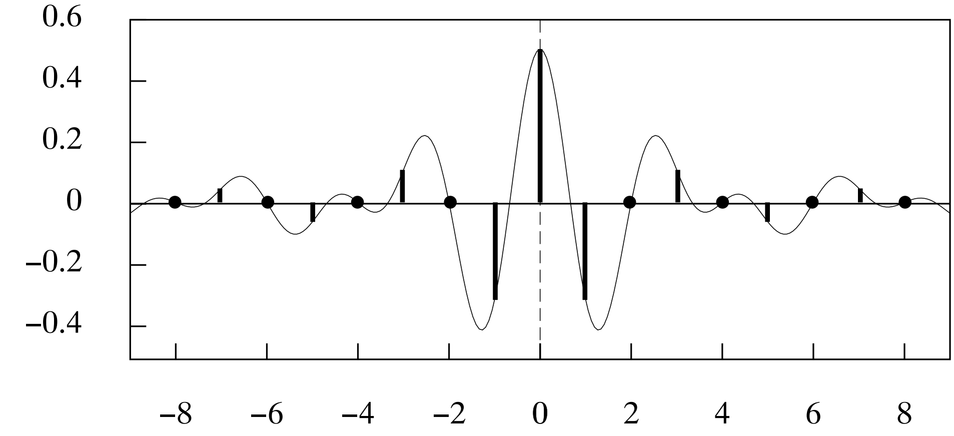

Figure 3. The wavelet function $\psi_{(1)}(t) = \cos(\pi t)\phi_{(0)}(t/2)$.

The Nyquist–Shannon sampling theorem indicates that

$$\phi_{1,0}(t) = \sum_{k=-\infty}^{\infty} g_k \frac{\sin\{\pi(t-k)\}}{\pi(t-k)} = \sum_{k=-\infty}^{\infty} g_k \phi(t-k), \tag{30}$$

where

$$g_k = \sqrt{2}\frac{\sin(\pi k/2)}{\pi t} = g_{-k} \tag{31}$$

are just the ordinates sampled at unit intervals from $\phi_{1,0}(t)$. These are also the coefficients of the filter function

$$G(z) = \{g_0 + g_1(z + z^{-1}) + g_2(z^2 + z^{-2)} + \cdots\}. \tag{32}$$

Equation (39) is nothing but the two-scale dilation equation for the sinc function bases.

The sinc function $\sin(\pi k/2)/\pi t$ is idempotent. It corresponds to a rectangle in the frequency domain of unit height defined the interval $[-\pi/2, \pi/2]$; and the squaring of the rectangle leaves it unaltered. The function is also symmetric, and it represents its own autocorrelation function. Thus, the conditions of sequential orthogonality amongst the basis functions $\phi_{1,k}(t) = \phi(2t-k)$ correspond to the zeros of the sinc function, which, as can be seen from Figure 2, are at two-point displacements.

A wavelet and a set of filter coefficients, to complement the scaling function of (29) and the filter coefficients of (31), may be obtained by the process of frequency shifting that translates the $\phi_{1,k}(t)$ in the upper frequency range $[\pi/2, \pi]$. The translation is affected by the function $\cos(\pi t) = (-1)^t$.

A one-unit time lag may also be imposed in fulfilment of the condition for lateral orthogonality that applies, in general, to symmetric IIR filters. (This is notwithstanding the fact that the scaling function and its frequency-shifted counterpart are already mutually orthogonal by virtue of their segregation within the frequency domain.) Thus, the wavelet may be specified by

$$\psi_{1,k}(t) = (-1)^t \phi_{1,k}(t-1) = \sqrt{2}\cos(\pi t)\frac{\sin(\pi t/2 - k - 1)}{\pi t}. \tag{33}$$

It is represented, together with its ordinates sampled at unit intervals, by Figure 3.

The corresponding filter coefficients, which are to be found in the dilation equation

$$\psi_{1,0}(t) = \sum_{k=-\infty}^{\infty} h_k \phi(t-k), \tag{34}$$

which are just the sampled ordinates of $\psi_{1,k}(t)$, are given by

$$h_k = (-1)^k g_{k-1}. \tag{35}$$

These are also the coefficients of the filter function

$$H(z) = zG(-z) = z\{g_0 - g_1(z + z^{-1}) + g_2(z^2 + z^{-2)} + \cdots\}. \tag{36}$$

Because its coefficients form a doubly-infinite sequence, the sinc function does not provide a practical filter. Also, the coefficients converge to zero slowly. The problems of an infinite sequence can be overcome by creating a circular filter of an order that is appropriate to the length of the data sequence.

An alternative way of adapting the filter is to truncate the sequence. Then, to obtain a desirable frequency response, which is not affected by ripples or by excessive leakage, it is appropriate to apply a taper to the coefficients via a symmetric sequence of weights $\{w_j; j = 0, \pm 1, \ldots, \pm M - 1\}$, with $w_j = w_{-j}$ and $w_0 = 1$.

The generic element of the weighted and truncated autocorrelation sequence may be denoted by

$$p_j = \begin{cases} w_j \dfrac{2\sin(\pi j/2)}{\pi j}, & \text{for } j = 0, \pm 1, \ldots, \pm M - 1; \\ 0, & \text{otherwise,} \end{cases} \tag{37}$$

Given that the sinc function already satisfies the conditions of sequential orthogonality, which is that the coefficients of $P(z)$ associated with even powers of z are zeros, it follows that, regardless of the choice of the weights, the weighted function will also satisfy the conditions.

For an appropriate weighting function, one might think of using the Blackman window (see Blackman and Tukey 1959) which is defined by

$$w_j = 0.42 + 0.5\cos\left(\frac{\pi j}{M}\right) + 0.08\cos\left(\frac{2\pi j}{M}\right), \qquad \text{where} \qquad |j| \leq M - 1. \tag{38}$$

An alternative weighting function is provided by the split cosine bell defined by

$$w_j = \begin{cases} 0.5\left[1 + \cos\dfrac{\pi(M + j)}{q}\right]; & j = 1 - M, \ldots, q - M, \\ 1.0; \quad j = q - M, \ldots, M - q, & \\ 0.5\left[1 + \cos\dfrac{\pi(q - M + j)}{q}\right]; & j = M - q, \ldots, M - 1. \end{cases} \tag{39}$$

This has a horizontal segment interpolated at the apex of the bell. Setting $q = M$ reduces this to the cosine bell.

If the autocorrelation function $P(z)$ is to be amenable to a spectral factorisation such that $P(z) = G(z)G(z^{-1})$, then it is necessary that $P(\omega) > 0$ for all $\omega \in [-\pi, \pi]$. If this is not the case and if $\min P(\omega) = q < 0$, then $P(z)$ can be replaced by $P(z) - q$. The autocorrelation function can be rescaled so as to satisfy the condition that $P(z) + P(-z) = 2$.

Once an appropriate positive-definite symmetric function is available, it can be factorised to give the function $G(z)$, which has an even number M of coefficients. Thereafter, the highpass filter $H(z) = -z^{M-1}G(-z^{-1})$ can be obtained by means of a signed reversal.

A numerical procedure for the spectral factorisation of the autocorrelation function has been provided by Tunnicliffe Wilson (1969) and it has been coded in Pascal and C by Pollock (1999). A discourse on the alternative methods for factorising a Laurent polynomial has been provided by Goodman *et al.* (1997).

Infinite Impulse Response Filters

One way of satisfying the condition of perfect reconstruction is to exploit the structure of the Wiener–Kolmogorov filters to derive a pair of complementary half-band filters. Let $F(z)$ be an arbitrary polynomial. Then, the autocorrelation function of the lowpass filter can take the form of

$$P(z) = \frac{2F(z)F(z^{-1})}{F(z)F(z^{-1}) + F(-z)F(-z^{-1})}; \tag{40}$$

and this may be factorised as $P(z) = G(z)G(z^{-1})$. It is easy to see that $P(z) + P(-z) = 2$, whereby the condition of perfect reconstruction is confirmed. The filter may be subject to an arbitrary number of translations in time that can be effected by an allpass filter or by a power of z. In that case, we may assume the factor affecting the translation is absorbed within the function $G(z)$. The corresponding highpass filter will be $H(z) = -z^{-1}G(-z)$.

A familiar example of such an autocorrelation function is provided by

$$\begin{aligned} P(z) &= \frac{2(1+z)^n(1+z^{-1})^n}{(1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n} \\ &= \frac{2}{1+\left(\mathrm{i}\dfrac{1-z}{1+z}\right)^{2n}}, \end{aligned} \tag{41}$$

which is the formula of the lowpass halfband Butterworth filter. The second expression is derived by dividing top and bottom of the first expression by the numerator. Then, the top and bottom of each factor within $\{(1-z^{-1})/(1+z^{-1})\}^n$ are multiplied by z. The factor i^{2n} provides n changes of sign.

The roots of $P(z)$, i.e. its poles and its zeros, come in reciprocal pairs; and, once they are available, they may be assigned unequivocally to the factors $G(z)$ and $G(z^{-1})$. Those roots which lie outside the unit circle belong to $G(z)$ whilst their reciprocals, which lie inside the unit circle, belong to $G(z^{-1})$.

The zeros of $P(z)$ are already available. To find the poles, consider the equation

$$(1+z)^n(1+z^{-1})^n + (1-z)^n(1-z^{-1})^n = 0, \tag{42}$$

which is equivalent to the equation

$$1+\left(\mathrm{i}\frac{1-z}{1+z}\right)^{2n}=0. \tag{43}$$

Solving the latter for

$$s=\mathrm{i}\frac{1-z}{1+z} \tag{44}$$

is a matter of finding the $2n$ roots of -1. These are given by

$$s=\exp\left\{\frac{\mathrm{i}\pi j}{2n}\right\},\quad \text{where}\quad j=1,3,5,\ldots,4n-1, \\ \text{or}\quad j=2k-1;\ k=1,\ldots,2n. \tag{45}$$

The roots correspond to a set of $2n$ points which are equally spaced around the circumference of the unit circle. The radii that join the points to the centre are separated by angles of π/n; and the first of the radii makes an angle of $\pi/(2n)$ with the horizontal real axis.

The inverse of the function $s=s(z)$ is the function

$$z=\frac{\mathrm{i}-s}{i+s}=\frac{\mathrm{i}(s-s^*)}{2-\mathrm{i}(s^*-s)}, \tag{46}$$

Here, the final expression comes from multiplying top and bottom of the second expression by $s^*-i=(i+s)^*$, where s^* denotes the conjugate of the complex number s, and from noting that $ss^*=1$. On substituting the expression for s from (34), it is found that the solutions of (34) are given, in terms of z, by

$$z_k=\mathrm{i}\frac{\cos\{\pi(2k-1)/2n\}}{1+\sin\{\pi(2k-1)/2n\}},\quad \text{where}\quad k=1,\ldots,2n. \tag{47}$$

The roots of $G(z^{-1})=0$ are generated when $k=1,\ldots,n$. Those of $G(z)=0$ are generated when $k=n+1,\ldots,2n$.

Given the availability of the analytic expressions for the roots of the Butterworth polynomial, we might hope to find a straightforward factorisation of the function $P(z)=G(z)G(z^{-1})$ that does not require an iterative procedure.

Herley and Vetterli (1993) have demonstrated that, in the special case where $F(z)$ is of an even length and when it comprises a symmetric sequence of coefficients, there is indeed a simple closed form factorisation of $P(z)$ that is available more generally to other versions of the Weiner–Kolmogorov function.

Consider, therefore, a causal filter

$$F(z)=f_0+f_1z+\cdots+f_1z^{N-1}+f_0z^N. \tag{48}$$

which has a even number $N+1$ of terms that form a symmetric sequence. Since the number of terms is even, there is no central point of symmetry within the sequence.

The terms associated with the even and the odd powers of z can be separated to form the polynomials

$$F_e(z^2) = f_0 + f_2 z^2 + \cdots + f_1 z^{N-1}, \qquad F_o(z^2) = f_1 + f_3 z^2 + \cdots + f_0 z^{N-1}, \quad (49)$$

for which $F_o(z^2) = z^{N-1}F_e(z^{-2})$. Then, it follows that

$$F(z) = F_e(z^2) + zF_o(z^2) = F_e(z^2) + z^N F_e(z^{-2}). \quad (50)$$

Therefore,

$$\begin{aligned} F(z)F(z^{-1}) &= \{F_e(z^2) + z^N F_e(z^{-2})\}\{F_e(z^{-2}) + z^{-N}F_e(z^2)\} \\ &= 2F_e(z^2)F_e(z^{-2}) \\ &\qquad + \{z^N F_e(z^2)F_e(z^{-2}) + z^{-N}F_e(z^2)F_e(z^{-2})\}. \end{aligned} \quad (51)$$

Since

$$F(z)F(z^{-1}) + F(-z)F(-z^{-1}) = \{F(z)F(z^{-1})\}_e \quad (52)$$

contains only even powers of z, and since N is an odd number, it follows that

$$\{F(z)F(z^{-1})\}_e = 2F_e(z^2)F_e(z^{-2}) \quad (53)$$

Therefore, the function of (31) can be expressed as

$$P(z) = \frac{F(z)F(z^{-1})}{F_e(z^2)F_e(z^{-2})} \quad (54)$$

of which the requisite factor is

$$G(z) = \frac{F(z)}{F_e(z^2)}. \quad (55)$$

The function that provides the frequency-domain profile of the Butterworth filter is obtained by setting $z = e^{-\mathrm{i}\omega}$ in (43). In that case,

$$1 + \left(\mathrm{i}\frac{1-z}{1+z}\right)^{2n} = 1 + \left(\mathrm{i}\frac{z^{-1/2} - z^{1/2}}{z^{-1/2} + z^{1/2}}\right)^{2n} = 1 + \left\{\frac{\sin(\omega/2)}{\cos(\omega/2)}\right\}^{2n}, \quad (56)$$

since $\sin(\omega/2) = -\mathrm{i}\{\exp(\mathrm{i}\omega/2) - \exp(-\mathrm{i}\omega/2)\}/2$ and $\cos(\omega/2) = \{\exp(\mathrm{i}\omega/2) + \exp(-\mathrm{i}\omega/2)\}/2$. Therefore, the function in question is

$$P(\omega) = \{1 + \tan(\omega)^{2n}\}^{-1}. \quad (57)$$

Filtering in the Frequency Domain

Within the time domain, filters are applied to the data via a process of convolution. If the data and the filter sequences are lengthy, then this may be a computationally demanding and a time consuming process. On the other hand, if the data sequence is short relative to the length of the filter, then there is liable to be a significant end-or-sample problem.

Provided that one is able to perform the computations off-line, then both problems can be addressed by performing the operations in the frequency domain. The time-consuming process of convolution in the time domain is converted into a more efficient process of modulation in the frequency domain, whereby the Fourier transforms of the data and the filter are multiplied together point by point.

The end-of-sample problem is handled automatically by the process of frequency-domain modulation, which corresponds to an application of circular convolution in the time domain. The latter would entail using the initial sample values as proxies for the values that lie beyond the end of the sample and using the final sample values as proxies for the presample values.

Provided that the data have been adequately detrended, there may be little harm in such a contrivance. What harm there might be can be mitigated by the provision of some carefully constructed synthetic data to be interpolated into the circular data sequence, where the end joins the beginning.

The essential condition that must be fulfilled by the frequency-domain versions of the wavelets filters is that of power complementarily whereby

$$P(\omega) + Q(\omega) = |G(\omega)|^2 + |H(\omega)|^2 = 2. \tag{58}$$

Moreover, if the functions $P(\omega)$and $Q(\omega)$ are to be mirror images of each other, then it must be that $Q(\omega) = P(\omega + \pi)$. The latter requires the functions to be reflections of each other about the about the vertical axis through $\pi/2$ and about the horizontal axis of unit height.

The condition

$$|G(\omega)|^2 + |G(\omega + \pi)|^2 = 2, \tag{59}$$

which comes from setting $H(\omega) = G(\omega + \pi)$ in (41) can also be deduced from the condition that

$$\sum_k |\phi(\omega + 2k\pi|^2 = 1. \tag{60}$$

As was indicated under (6.4), this is the frequency-domain equivalent of the condition othonormality affecting the family of scaling functions $\phi(t-k)$. The frequency-domain version of the dilation equation is

$$\phi(\omega) = 2^{-1/2} G(\omega/2)\phi(\omega/2), \tag{61}$$

and substituting this into (69) gives

$$\begin{aligned}
1 &= \frac{1}{2}\sum_k |G(\omega + k\pi)|^2 |\phi(\omega + k\pi)|^2 \\
&= \frac{1}{2}|G(\omega)|^2 \sum_k |\phi(\omega + 2k\pi)|^2 \\
&\qquad + \frac{1}{2}|G(\omega+\pi)|^2 \sum_k |\phi(\omega + [2k+1]\pi)|^2 \\
&= \frac{1}{2}\left\{|G(\omega)|^2 + |G(\omega+\pi)|^2\right\}
\end{aligned} \tag{62}$$

The condition of (60) is fulfilled, of course, by the Shannon filter defined under (29) which corresponds to a perfect rectangular halfband frequency response. However, this function has a slow hyperbolic rate of convergence as well as an infinite support in the time domain.

The difficulty of the infinite support can be overcome by the circular wrapping of the filter, which occurs when the frequency-domain rectangle is sampled at T points which are transformed via the inverse discrete Fourier transform to create circular filter.

The difficulty of the slow convergence of the Shannon wavelets is attributable to the sharp frequency cut-off at $\pi/2$. It can be addressed by imposing a more gradual transition from the pass band to the stop and vice versa.

There are numerous pairs of functions with more or less gradual transitions that satisfy the power complementary condition of (59). The simplest are derived by placing a cross at the points $\pm\pi/2$. The resulting power spectrum of the low pass filter, which may be described as a split triangle or as a chamfered box, is defined by

$$P(\omega) = \begin{cases} 1, & \text{if } |\omega| \in (0, \pi/2 - \epsilon), \\ 1 - \dfrac{|\omega + \epsilon - \pi/2|}{2\epsilon}, & \text{if } |\omega| = (\pi/2 - \epsilon, \pi/2 + \epsilon), \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

Setting $\epsilon = \pi/2$ reduces this to a triangular function. Also subsumed under this function is the rectangular function The discontinuity at the cut-off point is handled, in effect, by chamfering the edge. (When the edge of the box is chamfered in the slightest degree, the two function values at the point of discontinuity, which are zero and unity, will coincide at a value of one half.) The result can be a greatly improved rate of convergence; but this crude recourse fails achieve an optimal trade off between dispersion in the time domain and dispersion in the frequency domain,

A superior recourse is to use the Butterworth function of (57) to obtain a wide range of profiles for the power function $P(\omega)$. There is no reason why, in this context, the parameter n should be restricted to take an integer value. It will be found, for example that when $n = 0.65$ the Butterworth function provides a

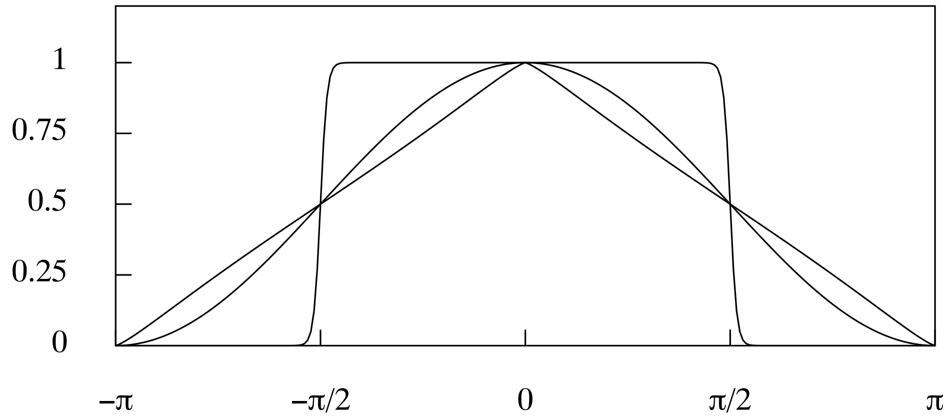

Figure 4. The Butterworth function with the parameter values $n = 0.62$ (the triangle), $n = 1$ (the bell) and $n = 20$ (the boxcar).

close approximation to the triangular power function. This is shown in Figure (4) together with the effects of other values of the parameter. The frequency response of the lowpass filter is obtained simply by taking the square root of $P(\omega)$,

To coefficient sequence corresponding to the function $G(\omega) = \sqrt{P}(\omega)$ would be obtained by subjecting it to the inverse of the discrete-time Fourier transform, which, in consequence of the symmetry of the function, becomes a cosine Fourier transform. Indeed, allowing for an interchange of time and frequency, this is nothing but a classical Fourier series transform.

However, the result from transforming $G(\omega)$ would be an infinite sequence of filter coefficients. To produce a circular filter of as many coefficients T as there are data elements, one should sample the function $G(\omega)$ at T equally-spaced points in the interval $[-\pi, \pi)$, or in the equivalent interval $[0, 2\pi)$. The filter coefficients would be obtained by applying the inverse discrete Fourier transform to these points.

Adapting the Filters to Finite Samples

The sequence of filter coefficients that corresponds to a power function defined in the frequency domain is liable to be infinite. Likewise, the corresponding wavelet is liable to require the entire real line for its support. However, in practice, a discrete wavelet analysis usually concerns a finite sample of T data points.

If T is very large or of indefinite length, as it is liable to be in the case of real-time or on-line processing, then this disparity can be ignored and a large number of coefficients, tending to zero with the increasing lags, can be comprised within a truncated filter sequence. In other cases, the disparity can be overcome by using circular versions of the data and of the filter.

In theory, a finite data sequence could be adapted to an infinite coefficient sequence by creating a periodic extension of the data in which the sample is replicated over every preceding and succeeding set of T integer points. which are $\{rT, rT+1, \ldots, (r+1)(T-1)\}$ with $r \in \{\pm 1, \pm 2, \ldots\}$.

By these means, the data value at a point $t \notin \{0, 1, \ldots, T-1\}$, which lies outside the sample, is provided by $y_t = y_{\{t \bmod T\}}$, which does lie within the sample. With the periodic extension available, one can think multiplying the filter coefficients point by point with the data and of shifting them any number of times.

As an alternative to extending the data, one can think of creating a finite sequence of filter coefficients by wrapping the infinite filter sequence $\{g_t\}$ around a circle of circumference T and adding the overlying coefficients to give

$$g_t^\circ = \sum_{k=-\infty}^{\infty} g_{\{t+kT\}} \quad \text{for} \quad t = 0, 1, \ldots, T-1. \tag{64}$$

The inner product of the resulting coefficients $g_0^\circ, \ldots, g_{T-1}^\circ$ with a finite sequence $x_0, \ldots, x_{T-1}$ will be identical to that of the original coefficients with the extended sequence. To show this, let $\tilde{x}(t) = \{\tilde{x}_t = x_{\{t \bmod T\}}\}$ denote the infinite sequence that is the periodic extension of $x_0, \ldots, x_{T-1}$. Then

$$\begin{aligned} \sum_{t=-\infty}^{\infty} g_t \tilde{x}_t &= \sum_{k=-\infty}^{\infty} \left\{ \sum_{t=0}^{T-1} g_{\{t+kT\}} \tilde{x}_{\{t+kT\}} \right\} \\ &= \sum_{t=0}^{T-1} x_t \left\{ \sum_{k=-\infty}^{\infty} g_{\{t+kT\}} \right\} = \sum_{t=0}^{T-1} g_t^\circ x_t. \end{aligned} \tag{65}$$

Here, the first equality, which is the result of cutting the sequence $\{g_t \tilde{x}_t\}$ into segments of length T, is true in any circumstance, whilst the second equality uses the fact that $\tilde{x}_{\{t+kT\}} = x_{\{t \bmod T\}} = x_t$. The final equality invokes the definition of g_t°.

In fact, the process of wrapping the filter coefficients should be conducted in the frequency domain, where it is simple and efficient, rather than in the the time domain, where it entails the summation of an infinite sequence. We shall elucidate these matters while demonstrating the use of the discrete Fourier transform in performing a wavelet analysis.

To elucidate these matters, consider the z-transforms of the filter sequence and the data sequence:

$$G(z) = \sum_{t=-\infty}^{\infty} g_t z^t \qquad \text{and} \qquad x(z) = \sum_{t=0}^{T-1} x_t z^t. \tag{66}$$

Setting $z = \exp\{-\mathrm{i}\omega\}$ in $G(z)$ creates a periodic function in the frequency domain of period 2π, denoted by $g(\omega)$, which, by virtue of the discrete-time Fourier transform, corresponds one-to-one with the doubly infinite time-domain sequence of filter coefficients.

Setting $z = z_j = \exp\{-\mathrm{i}2\pi j/T\}; j = 0, 1, \ldots, T-1$, is tantamount to sampling the continuous function $G(\omega)$ at T points within the frequency range of $\omega \in [0, 2\pi)$.

(Given that the data sample is defined on a set of positive integers, it is appropriate to replace the symmetric interval $[-\pi, \pi]$, considered hitherto, in which the endpoints are associated with half the values of their ordinates, by the positive frequency interval $[0, 2\pi)$, which excludes the endpoint on the right and attributes the full value of the ordinate at zero frequency to the left endpoint.) The powers of z_j now form a T-periodic sequence, with the result that

$$\begin{aligned} G(z_j) &= \sum_{t=-\infty}^{\infty} g_t z_j^t \\ &= \left\{ \sum_{k=-\infty}^{\infty} g_{kT} \right\} + \left\{ \sum_{k=-\infty}^{\infty} g_{(kT+1)} \right\} z_j + \cdots + \left\{ \sum_{k=-\infty}^{\infty} g_{(kT+T-1)} \right\} z_j^{T-1} \\ &= g_0^\circ + g_1^\circ z_j + \cdots + g_{T-1}^\circ z_j^{T-1} = G^\circ(z_j). \end{aligned} \tag{67}$$

There is now a one-to-one correspondence, via the discrete Fourier transform, between the values $G(z_j); j = 0, 1, \ldots, T-1$, sampled from $G(\omega)$ at intervals of $2\pi/T$, and the coefficients $g_0^\circ, \ldots, g_{T-1}^\circ$ of the circular wrapping of $g(t)$. Setting $z = z_j = \exp\{-\mathrm{i}2\pi j/T\}; j = 0, 1, \ldots, T-1$, within $y(z)$ creates the discrete Fourier transform of the data sequence, which is commensurate with the square roots of the ordinates sampled from the energy function.

The Daubechie Maxflat FIR Filters

The filters that have come to dominate dyadic wavelets analysis are the ones that have been proposed by Daubechies (1988, 1992). These are the so-called maxflat halfband FIR filters that entail an even number $M = 2m$ of coefficients of which z-transforms constitute polynomials of degree $M - 1$. The lowpass scaling function filter $G(z)$ and the highpass wavelets filter $H(z)$ form a power complementary pair of which the sum of the squared gain functions is a constant function:

$$G(z)G(z^{-1}) + H(z)H(z^{-1}) = 2 \tag{68}$$

A maxflat condition is fulfilled when there is a maximum number of zero-valued derivatives at a specific point or set of points in the frequency response. The condition that is imposed on the lowpass filter, of which the z-transform is $G(z) = g_0 + g_1 z + \cdots + g_{M-1} z^{M-1}$, is that the response has the maximum number of zeros at the point $z = -1$, which correspond to factors of $1 + z$ within $G(z)$.

Once the lowpass filter has been specified, the condition of sequential orthogonality requires that the highpass filter should be

$$H(z) = -z^{M-1} G(-z^{-1}). \tag{69}$$

Therefore, the maxflat condition affecting $G(z)$ imposes the same number of zeros on $H(z)$ at the point $z = 1$, which correspond to factors of $1 - z$. Given that the two filters are complementary, the two sets of maxflat conditions imply that the two filters have flat frequency responses both at $z = 1$ and at $z = -1$.

The filter $G(z)$ is derived by factorising the autocovariance function $P(z) = G(z)G(z^{-1})$, which has $4m-1$ coefficients associated with powers of z ranging from $1-2m$ to $2m-1$.

The conditions of sequential orthogonality require that $P(z)$ has $2m-2$ zero-valued coefficients: $p_{2j} = 0; j = \pm 1, \ldots, \pm(m-1)$. There is also a central coefficient with the value of $p_0 = 1$. This leaves a reminder of $2m$ coefficients that can be used in placing zeros in $P(z)$ at $\omega = \pi$, which correspond to polynomial roots at $z = z^{-1} = \exp\{\pm i\pi\} = -1$. In that case, the autocovariance function must take the form of

$$P(z) = G(z)G(z^{-1}) = \left(\frac{1+z}{2}\right)^m W(z)\left(\frac{1+z^{-1}}{2}\right)^m, \tag{70}$$

where $W(z) = W(z^{-1})$ is a symmetric polynomial of $2m-1$ coefficients, associated with powers of z running from $1-m$ to $m-1$. This can be factorised as $W(z) = V(z)V(z^{-1})$, whereafter $G(z) = \{(1+z)/2\}^m V(z)$ can be formed.

Observe that the presence of the operator $(1-z)^m$ within $H(z)$ implies the this filter will nullify the ordinates of polynomial of degree $m-1$. Therefore, the condition of perfect reproduction, which is a feature of an orthogonal filter bank, implies that $G(z)$ will transmit the ordinates of the polynomial.

The factors of $W(z)$ can be obtained via an iterative procedure, but, in some simple cases, it is possible to perform the factorisation analytically as the following example shows, which concerns the Daubechies D4 filter. This is a filter of length 4 that satisfies the conditions of sequential and lateral orthogonality.

Example. Let $m = 2$ and let $W(z) = \alpha z^{-1} + \beta + \alpha z$. On compounding this with the factors $\{(1+z)/2\}^2 = \{1 + 2z + z^2\}/4$ and $\{(1-z)/2\}^2$, we get

$$\begin{aligned} P(z) = \{\alpha z^{-3} &+ (4\alpha+\beta)z^{-2} + (7\alpha+4\beta)z^{-1} + (8\alpha+6\beta) \\ &+ (7\alpha+4\beta)z + (4\alpha+\beta)z^2 + \alpha z^{-3}\}/16. \end{aligned} \tag{71}$$

The conditions of sequential orthogonality indicate that the coefficients associated with z^2 and z^{-2} are zeros. The coefficient associated with z^0 is unity. Therefore,

$$4\alpha + \beta = 0 \qquad \text{and} \qquad 8\alpha + 6\beta = 16. \tag{72}$$

The solutions of these equations are

$$\alpha = -1 \qquad \text{and} \qquad \beta = 4, \tag{73}$$

and $W(z) = V(z)V(z^{-1})$ becomes

$$\begin{aligned} W(z) &= -z^{-1} + 4 - z \\ &= \frac{1}{2}(\{1+\sqrt{3}\} + \{1-\sqrt{3}\}z^{-1})(\{1+\sqrt{3}\} + \{1-\sqrt{3}\}z). \end{aligned} \tag{74}$$

It follows that the lowpass filter $G(z) = \{(1+z)/2\}^2 V(z)$ is given by

$$\begin{aligned} G(z) &= \left(\frac{1}{4\sqrt{2}}\right)(1+z)^2(\{1+\sqrt{3}\} + \{1-\sqrt{3}\}z) \\ &= \left(\frac{1}{4\sqrt{2}}\right)(\{1+\sqrt{3}\} + \{3+\sqrt{3}\}z + \{3-\sqrt{3}\}z^2 + \{1-\sqrt{3}\}z^3). \end{aligned} \tag{75}$$

The Method of Daubechies

The original approach pursued by Daubechies in deriving maxflat filter of higher orders was somewhat complicated. It has the virtue, nevertheless, of identifying the functional form, in general, of the polynomial $V(z)$ within $G(z) = \{(1+z)/2\}^m V(z)$.

To begin, one may consider the expressions for $P(z) = G(z)G(z^{-1})$ and $P(-z)$ that incorporate the zeros at $z = -1$ and at $z = 1$ respectively. These are

$$\begin{aligned} P(z) &= \left(\frac{1+z}{2}\right)^m W(z) \left(\frac{1+z^{-1}}{2}\right)^m \qquad \text{and} \\ P(-z) &= \left(\frac{1-z}{2}\right)^m W(-z) \left(\frac{1-z^{-1}}{2}\right)^m. \end{aligned} \tag{76}$$

Setting $z = \exp\{-\mathrm{i}\omega\}$ within

$$\left(\frac{1+z}{2}\right)\left(\frac{1+z^{-1}}{2}\right) = \frac{1}{2}\left\{1 + \frac{z+z^{-1}}{2}\right\} = \left\{\frac{z^{1/2}+z^{-1/2}}{2}\right\}^2 \tag{77}$$

gives

$$\frac{1+\cos(\omega)}{2} = \cos^2(\omega/2) = 1 - y. \tag{78}$$

Replacing z in (77) by $-z$ and again setting $z = \exp\{-\mathrm{i}\omega\}$ gives

$$\frac{1-\cos(\omega)}{2} = \sin^2(\omega/2) = y. \tag{79}$$

Therefore, the condition for sequential orthogonality, which is that $P(z)+P(-z) = 2$, can be expressed as

$$\begin{aligned} 2 &= P(\omega) + P(\omega+\pi) \\ &= \left\{\cos^2\left(\frac{\omega}{2}\right)\right\}^m W(\omega) + \left\{\sin^2\left(\frac{\omega}{2}\right)\right\}^m W(\omega+\pi). \end{aligned} \tag{80}$$

Next, it is recognised that the functions $W(z)$ and $W(-z)$ with $z = \exp\{-\mathrm{i}\omega\}$ can be expressed as trigonometrical polynomials:

$$W(\omega) = Q(\sin^2\{\omega/2\}), \qquad W(\omega+\pi) = Q(\cos^2\{\omega/2\}). \tag{81}$$

Therefore, on setting $\sin^2\{\omega/2) = y$ and $\cos^2\{\omega/2) = 1 - y$, equation (80) can be written as

$$2 = (1 - y)^m Q(y) + y^m Q(1 - y). \tag{82}$$

Now, the object is to find a solution to the polynomial $Q(y)$, which will lead to $W(z) = V(z)V(z^{-1})$ and thence to $G(z)$. To this end, it is appropriate to consider the equation

$$\begin{aligned}
1 &= \{(1 - y) + y\}^{2m-1}\\
&= \sum_{j=0}^{2m-1} \binom{2m-1}{j}(1 - y)^j y^{2m-1-j}\\
&= \sum_{j=0}^{m-1} \binom{2m-1}{j}(1 - y)^j y^{2m-1-j} + \sum_{j=m}^{2m-1} \binom{2m-1}{j}(1 - y)^j y^{2m-1-j}.
\end{aligned} \tag{83}$$

Using

$$\binom{2m-1}{j} = \binom{2m-1}{2m-1-j} = \frac{(2m-1)!}{j!(2m-1-j)!}$$

and defining $k = 2m - 1 - j$ enables us to rewrite the second term of (91) as

$$\sum_{k=0}^{m-1} \binom{2m-1}{k} y^k (1 - y)^{2m-1-k}, \tag{84}$$

whence, on multiplying by 2, equation (83) becomes

$$\begin{aligned}
2 =& y^m 2 \sum_{j=0}^{m-1} \binom{2m-1}{j}(1 - y)^j y^{m-1-j}\\
&\quad + (1 - y)^m 2 \sum_{j=0}^{m-1} \binom{2m-1}{j} y^j (1 - y)^{m-1-j}\\
=& y^m Q(1 - y) + (1 - y)^m Q(y),
\end{aligned} \tag{85}$$

where

$$Q(y) = 2 \sum_{j=0}^{m-1} \binom{2m-1}{j} y^j (1 - y)^{m-1-j}. \tag{86}$$

Here, we may observe that $Q(y)$ is a polynomial of degree $m - 1$, which may be indicated by denoting it by $Q_{m-1}(y)$. It is straightforward to show that

$$\begin{aligned}
\tfrac{1}{2}Q_0(y) &= 1,\\
\tfrac{1}{2}Q_1(y) &= 1 + 2y,\\
\tfrac{1}{2}Q_2(y) &= 1 + 3y + 6y^2,\\
\tfrac{1}{2}Q_4(y) &= 1 + 4y + 10y^2 + 20y^3.
\end{aligned} \tag{87}$$

An easy means of generating such coefficients is illustrated by the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}. \tag{88}$$

Here, as one moves from left to right, the elements of each column after the first are formed via a running total of the elements of the previous column. Also, the successive matrix diagonals of a NE–SW orientation contain the coefficients from successive rows of Pascal's triangle.

An alternative form for $Q(y)$ can be found by considering the matter of solving to equation (82) directly. The solution must satisfy the equation

$$Q(y) = (1-y)^{-m}\{2 - y^m Q(1-y)\}, \tag{89}$$

and, given that it is a polynomial of degree $m-1$, this is bound to comprise the first m terms of the expansion of $(1-y)^{-m}$. The higher order terms of the expansion will be cancelled with terms within $y^m Q(1-y)$. The coefficient of y^k within the Taylor series or binomial expansion of $(1-y)^{-m}$ is

$$\frac{m(m+1)\cdots(m+k-1)}{k!} = \frac{(m+k-1)!}{k!(m-1)} = \binom{m+k-1}{k}. \tag{90}$$

Therefore, the alternative expression for the solution is

$$Q(y) = 2\sum_{k=0}^{m-1}\binom{m+k-1}{k}y^k. \tag{91}$$

It is easy to recognise that this also generates the equations of (87) and that the coefficients of the expansion of $(1-y)^{-m}$ are the elements of the mth column of an indefinitely extended version of the matrix of (87).

Now, by setting $y = \sin^2(\omega/2)$ and $1 - y = \cos^2(\omega/2)$ within the equation $(1-y)^m Q(y)$ and using the identities of (78) and (79), it can be seen that

$$P(\omega) = 2\left(\frac{1+\cos(\omega)}{2}\right)^m \sum_{k=0}^{m-1}\binom{m+k-1}{k}\left(\frac{1-\cos(\omega)}{2}\right)^k, \tag{92}$$

which can also be rendered as

$$P(\omega) = 2\left(\frac{1+z}{2}\right)^m\left(\frac{1+z^{-1}}{2}\right)^m \sum_{k=0}^{m-1}\binom{m+k-1}{k}\left(\frac{1+z}{2}\right)^k\left(\frac{1+z^{-1}}{2}\right)^k. \tag{93}$$

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