Two-Channel Filter Banks and Dyadic Decompositions

by D.S.G. Pollock
University of Leicester
Email: stephen_pollock@sigmapi.u-net.com

In the basic subband coding procedure, which has been used in the transmission of speech via digital media, the signal is first split between two frequency bands via complimentary highpass and lowpass filters and then it is subsampled and encoded for transmission. At the receiving end, the signals of the two channels are decoded and then upsampled, by the interpolation of zeros between the sample elements. Then, they are smoothed by filtering, before they are recombined in order to provide a representation the original signal. The objective here is a perfect reconstruction of the original signal.

In a wavelets analysis, there is a further requirement, which is that the basis functions or wavelets, in terms of which the continuous signal may be expressed, should be mutually orthogonal both sequentially, which is within the channels, and laterally, which is across the channels. The orthogonality of the wavelets is guaranteed if the corresponding filters, which comprise the coefficients of the wavelets dilation equations, are mutually orthogonal in both ways.

In this chapter, we shall derive the conditions for perfect reconstruction, which place certain restrictions on the filter coefficients. Then, we shall proceed to derive the more stringent conditions of orthogonality. First, we must consider the basic procedures for dividing the signal into a high-frequency and a low-frequency component and for transmitting and recombining these components.

Once the architecture of the two-channel filter bank has been established, it can be used in pursuit of a dyadic decomposition of the frequency range in which, in descending the frequency scale, successive frequency bands have half the width of their predecessors. Thereafter, we may consider dividing the frequency range into 2^n bands of equal width.

Subband Coding and Transmission

The path taken by the signal through the highpass branch of the two-channel network may be denoted by

$$y(z) \longrightarrow H(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow E(z) \longrightarrow w(z), \tag{1}$$

whereas the path taken through the lowpass branch may be denoted by

$$y(z) \longrightarrow G(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow D(z) \longrightarrow v(z). \tag{2}$$

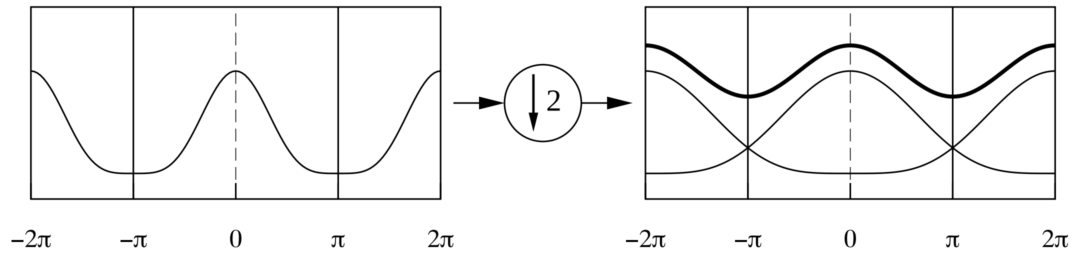

Figure 1. The effects in the frequency-domain of downsampling by a factor of 2. On the left is the original spectral function and on the right is its dilated version together with a copy displaced by 2π radians. To show the effects of aliasing, the ordinates of the two functions must be added to produce the bold line.

Here, $G(z)$ and $H(z)$ may be described as the analysis filters, and $D(z)$ and $E(z)$ may be described as the synthesis filters. The symbol $(\downarrow 2)$ denotes the operation of downsampling, which entails the deletion of the signal elements with odd-values indices. The symbol $(\uparrow 2)$ denotes the operation of upsampling, which inserts zeros between adjacent elements. The symbol $\simeq$ denotes the storage and transmission of the signals. The output signal, formed by merging the two branches, is $x(t) = v(t) + w(t)$.

The immediate objective is to find the z-transform of the output signal $x(t)$. Thereafter, we may seek the conditions under which the output signal is a reconstruction of the input signal $y(t)$, with possible variations of amplitude and phase.

Consider any absolutely summable discrete-time signal $p(t)$ that has been subject to the processes of downsampling and upsampling to produce the sequence $q(t) = p\{(t \downarrow 2) \uparrow 2\}$. This may be described as the alternated version of the sequence $p(t)$, in recognition of the fact that alternate elements have been replaced by zeros.

Let $p(t) \longleftrightarrow p(\omega)$, which is to say that $p(\omega)$ is the Fourier transform or "spectrum" of $p(t)$. Since $p(\omega)$ is a 2π-periodic function, we may suppose that its domain is a circle, in which case ω may be confined to an interval of 2π radians. The choice of the range of ω, whether it be $\omega \in [0, 2\pi]$ or $\omega \in [-\pi, \pi]$, is then only a matter of convenience.

In the process of downsampling, the argument ω of $p(\omega)$ is replaced by $\omega/2$, and the frequency-domain function evolves at half the previous rate, so that it accomplishes one cycle as ω runs from zero to 4π or from -2π to 2π. Therefore, the function is wrapped twice around the circumference of the circle, and the overlying ordinates are added. The effect is one of spectral aliasing.

This interpretation of the effects of downsampling is represented by the central segments of the two sides of Figure 1 that are supported on the interval $[-\pi, \pi]$. On the left is the original spectral function and on the right is its dilated version of which the wrapped tails are superimposed the central segment. The aliased function is the result of the addition of the overlapping functions on the right.

The diagram also supports an alternative interpretation in which the periodic trigonometric functions are defined on the real line. Then, the downsampling entails the superimposition of a copy of the dilated function at a

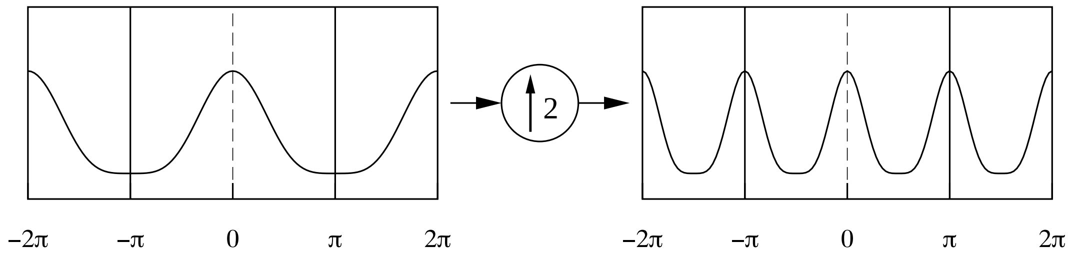

Figure 2. The effects in the frequency-domain of upsampling by a factor of 2. On the left is the original spectral function. On the right, the rate at which the frequency function evolves has been doubled and the function undergoes two cycles as ω traverses an interval of 2π radians .

displacement of 2π radians.

In the process of upsampling by a factor of 2, the frequency argument is multiplied by 2 and the rate at which the frequency function evolves is doubled. Thus, the function undergoes two cycles as the argument ω traverses an interval of 2π radians, and two images of the spectrum are mapped into the interval. This imaging effect of upsampling can be seen in Figure 2.

The effect of downsampling is summarised by writing $p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(\omega/2) + p(\pi + \omega/2)$. Since $\exp\{\pm i\pi\} = -1$ and $\exp\{-i(\pi + \omega/2)\} = -\exp\{-i\omega/2\}$, this can be expressed, in terms of the z-transform, wherein $z = \exp\{-i\omega\}$, as

$$p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(z^{1/2}) + p(-z^{1/2})\}. \tag{3}$$

The effect of the subsequent upsampling, which doubles the value of the frequency argument, is summarised by $p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(\omega) + p(\pi + \omega)\}$, which can also be written as

$$p\{(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(z) + p(-z)\}. \tag{4}$$

It follows that the signals that emerge from the two branches of the network depicted in (1) and (2) are given by

$$\begin{aligned} w(z) &= \frac{1}{2}E(z)\{H(z)y(z) + H(-z)y(-z)\}, \\ v(z) &= \frac{1}{2}D(z)\{G(z)y(z) + G(-z)y(-z)\}. \end{aligned} \tag{5}$$

Adding the two signals gives

$$\begin{aligned} x(z) = \frac{1}{2}&\{D(z)G(-z) + E(z)H(-z)\}y(-z), \\ &+ \frac{1}{2}\{D(z)G(z) + E(z)H(z)\}y(z). \end{aligned} \tag{6}$$

In matrix terms, this can be written as

$$\begin{aligned} x(z) &= v(z) + w(z) \\ &= \frac{1}{2}\begin{bmatrix} D(z) & E(z) \end{bmatrix} \begin{bmatrix} G(z) & G(-z) \\ H(z) & H(-z) \end{bmatrix} \begin{bmatrix} y(z) \\ y(-z) \end{bmatrix}. \end{aligned} \tag{7}$$

Conditions for Perfect Reconstruction

A perfect reconstruction of the input signal, without any ostensible processing delay, can be achieved if the term in $y(-z)$, which is due to aliasing and imaging, is eliminated and if the term in $y(z)$ contains an identity transformation:

$$D(z)G(z) + E(z)H(z) = 2, \tag{8}$$

$$D(z)G(-z) + E(z)H(-z) = 0. \tag{9}$$

Since the LHS of equation (9) is associated with the term $y(-z)$, it represents the upsampled aliasing effects. The aliasing effects within the two branches, if they are present, can be eliminated only by virtue of their mutual cancellation. There would be no such effects if it were possible too implement the ideal half-band filters.

To simplify the illustration, let it be assumed that $G(z) = G(z^{-1})$ and $H(z) = H(z^{-1})$ are the ideal and symmetric halfband filters, for which $H(z) = G(-z)$, $G(z) = H(-z)$ and $G(z^{-1})H(z) = 0$. That is to say, the filters are mirror images of each other, they are mutually orthogonal, and they have no spectral overlap. Then, the appropriate choice of the synthesis filters would be the reversed filters $D(z) = G(z^{-1})$ and $E(z) = H(z^{-1})$ which, in view of the symmetry, are none other than $G(z)$and $H(z)$ respectively.

In that case, it is clear that equation (8), which becomes

$$G(z^{-1})G(z) + H(z^{-1})H(z) = 2, \tag{10}$$

would be satisfied provided that the gains of the filters are adjusted accordingly, since, according to the usual definition, the ideal filters have unit gain throughout their pass bands and zero gain within their stop bands. Since $D(z)G(-z) = G(z^{-1})H(z) = 0$ and $E(z)H(-z) = H(z^{-1})G(z) = 0$, equation (9) would also be satisfied.

The roles of the ideal halfband filters within the two-channnel structure can be described in a manner that clearly reveals the essential effects of the network. For the purposes of a graphical illustration, matters are simplified by considering a symmetric real-valued time-domain signal, which has a real-valued symmetric spectrum. Then, there is no imaginary component to contend with.

Consider first the lowpass branch, represented by (2) and illustrated in the upper tranche of Figure 3. The effect of the filter $G(z)$ will be to preserve the contents of the signal that lie in the positive low-frequency half-interval $[0, \pi/2)$ and in the complementary interval $[-\pi/2, 0)$ and to eliminate the contents that lie in the high-frequency half-interval $[\pi/2, \pi)$ and its complement $[-\pi, -\pi/2)$. The downsampling operation that follows will spread the contents of the central low-frequency interval $[-\pi/2, \pi/2)$ across the full spectral range of $[-\pi, \pi)$.

On entering the synthesis stage of the branch, represented in the lower tranche of Figure 3, the signal is upsampled. This has the effect of compressing the dilated spectral image into the central interval $[-\pi/2, \pi/2)$ and of replicating the compressed image over the upper interval $[\pi/2, \pi) \cup [-\pi, -\pi/2)$. The

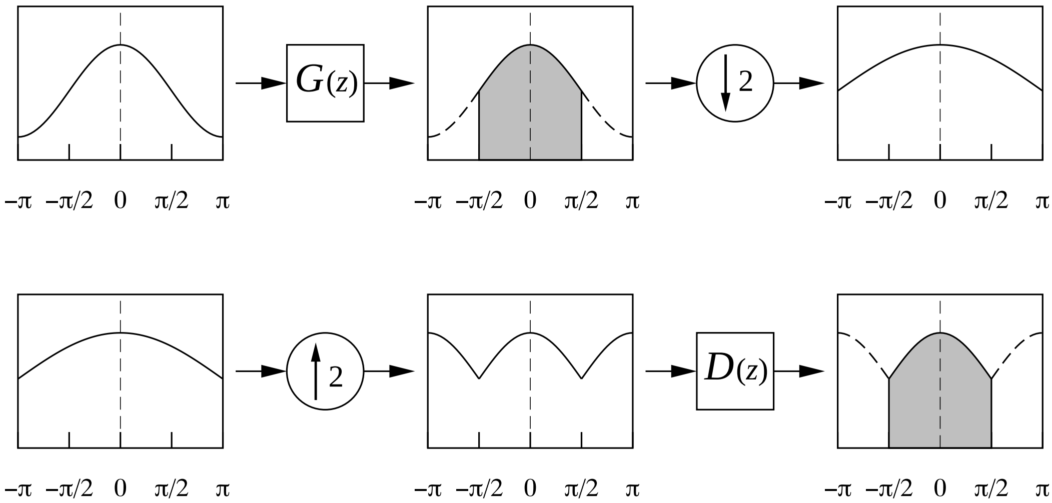

Figure 3. The effects on the spectrum of the lowpass branch of the two-channel filter bank, in the case where $G(z)$ and $D(z) = G(z)$ are ideal halfband filters.

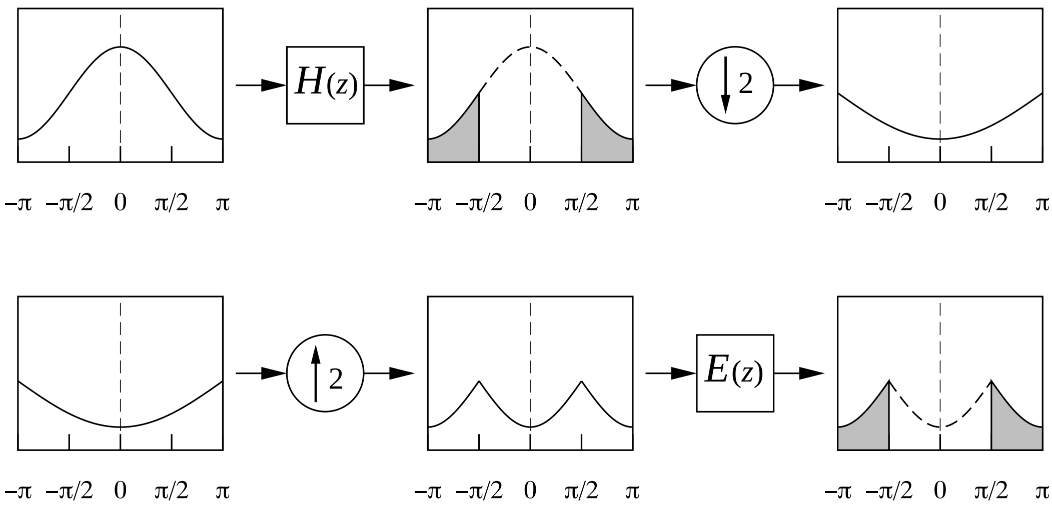

Figure 4. The effects on the spectrum of the highpass branch of the two-channel filter bank, in the case where $H(z)$ and $E(z) = H(z)$ are ideal halfband filters.

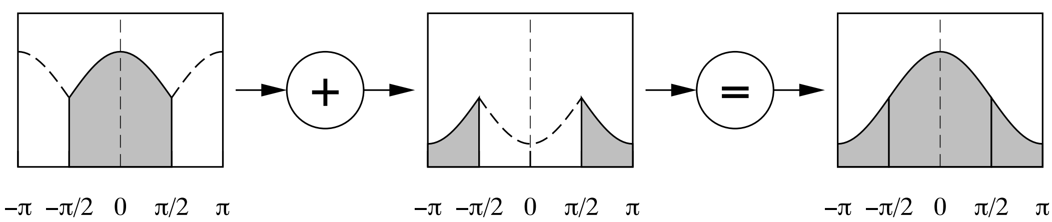

Figure 5. The perfect reconstruction of the signal spectrum via the addition of the lowpass spectrum and the highpasss spectrum, in the case where the two-channel filter bank incorporates the ideal halfband filters.

effect of the filter $D(z) = G(z^{-1})$ will be to remove the upper image. What remains is the lower half of the original spectral content, contained within $w(z)$.

The highpass branch is represented by (1) and illustrated in Figure 4. It begins with the filter $H(z)$, which has the effect of isolating the contents of the signal that lie in the upper interval. Then, the successive operations of downsampling and upsampling first dilate the spectral image and then compress it and replicate it over the two half intervals.

The image that lies in the lower half-interval is removed by the filter $E(z) = H(z^{-1})$. The upper half of the original spectral image is, therefore, contained within $v(z)$. By adding $v(z)$ and $w(z)$, the original signal is recovered, albeit with an imposed delay. The reconstruction of the signal's spectrum is illustrated in Figure 5.

The ideal filters are available only when it is possible to perform the calculations offline. However, early attempts at signal reconstruction relied on approximations to the ideal filters.

Allowing for the interchange of z and $-z$, equations (8) and (9) can be rendered, in matrix terms, as

$$\frac{1}{2}\begin{bmatrix} D(z) & E(z) \\ D(-z) & E(-z) \end{bmatrix}\begin{bmatrix} G(z) & G(-z) \\ H(z) & H(-z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11}$$

This condition of perfect reconstruction is described, equivalently, as the condition of biorthogonality.

Now, it is necessary to solve for $D(z)$ and $E(z)$ given $G(z)$ and $H(z)$. The equation to be solved is

$$\frac{1}{2}\,[\,D(z) \quad E(z)\,]\begin{bmatrix} G(z) & G(-z) \\ H(z) & H(-z) \end{bmatrix} = [\,1 \quad 0\,]. \tag{12}$$

This gives

$$\begin{aligned}\begin{bmatrix} D(z) \\ E(z) \end{bmatrix} &= \begin{bmatrix} G(z) & H(z) \\ G(-z) & H(-z) \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{2}{\Delta(z)}\begin{bmatrix} H(-z) & -H(z) \\ -G(-z) & G(z) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix},\end{aligned} \tag{13}$$

where

$$\Delta(z) = G(z)H(-z) - H(z)G(-z) = -\Delta(-z) \tag{14}$$

is a determinant. Thus

$$D(z) = \frac{2}{\Delta(z)}H(-z) \qquad \text{and} \qquad E(z) = \frac{-2}{\Delta(z)}G(-z). \tag{15}$$

A set of filters that fulfil these conditions will guarantee the perfect reconstruction of the signal, in the absence of any errors of quantisation.

The present prescriptions are incomplete, since the filters of the analysis section, which are the lowpass $G(z)$ and the highpass filter and $H(z)$, are yet to be specified in full. One way in which this can be achieved is by asserting the conditions of orthogonality.

Quadrature Mirror Filters

The possibility of achieving perfect signal reconstruction via a two-channel filter bank in which there is a degree of spectral overlap between the channels was first recognised by Croisier, Esterban and and Galand (1976) and later reiterated by Esterban and and Galand (1977). They proposed to use so-called quadrature mirror filters. These are pairs of filters of which the frequency response of one filter is the mirror image, about the value $\pi/2$, of that of the other filter. Thus $H(z) = G(-z)$. The additional specifications were

$$D(z) = G(z) \quad \text{and} \quad E(z) = -H(z) = -G(-z). \tag{16}$$

It can be see that the condition of (9) for the cancellation of the aliasing effects arising from the downsampling, which is that $D(z)G(-z)+E(z)H(-z) = 0$, is satisfied, since it becomes $G(z)G(-z) + \{-G(-z)\}\{G(z)\} = 0$. The other condition (8) of perfect reconstruction, which, with the allowance of a lag, requires that $D(z)G(z) + E(z)H(z) = 2z^q$, for some integer lag value q, will be satisfied if the filters can be chosen such that

$$G(z)G(z) - G(-z)G(-z) = G(z)G(z) - H(z)H(z) = 2z^q. \tag{17}$$

This is hard to achieve. In fact, for FIR filters, the condition cannot be satisfied exactly except by the Haar filter. For the causal Haar filter $G(z) = (1+z)/\sqrt{2}$, the condition becomes

$$\frac{1}{2}(1 + 2z + 2z^2) - (1 - 2z + 2z^2) = 2z. \tag{18}$$

Filters that closely approximate the condition of (17) must have a rapid transition from the passband to the stopband as well as high stopband attenuation. A method of designing such filters has been described by Johnston (1980).

Conditions of Orthogonality

The conditions of prefect reconstruction impose weak restrictions on the choice of the filters that are to be incorporated in the two-channel network. Additional conditions may also be imposed that will narrow the choice. The most common additional restrictions are those of orthogonality. These allow the data sequence to be expressed in terms of an orthogonal basis consisting of replicas of the sequences of filter coefficients at successive displacements.

In the case of the two-channel filter bank, where there is downsampling by a factor of two, the displacements are by two points at a time. This will ensure that there are as many basis functions as there are data points,

The orthogonality of the filter sequences within either channel may be described as *sequential* orthogonality. There is also orthogonality between the coefficient sequences in either channel. This may be described as *lateral* orthogonality. The orthogonality of the filter coefficients will give rise to the orthogonality of the corresponding wavelets, which are functions in continuous time.

The conditions of sequential orthogonality are expressed in terms of the autocorrelation functions of the filters. These are $G(z)G(z^{-1})$, in the case of the lowpass filter and $H(z)H(z^{-1})$, in case of the highpass filter. The sequential orthogonality at displacements that are multiples of two points implies that the alternated versions of the autocovariance sequences must be zeros-valued, except for the nonzero central coefficients, to which a value of 2 may be assigned. (The alternated versions of the sequences are produced by downsampling and upsampling the sequences in succession.)

Thus, the conditions of sequential orthogonlity, expressed in terms of the z-transforms, are

$$G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2 \qquad \textit{Sequential Orthogonality} \tag{19}$$

and

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2. \qquad \textit{Sequential Orthogonality} \tag{20}$$

The conditions of lateral orthogonality are expressed, likewise, in terms of the cross correlation function of the filters:

$$G(z)H(z^{-1}) + G(-z)H(-z^{-1}) = 0. \qquad \textit{Lateral Orthogonality} \tag{21}$$

The conditions of lateral and sequential orthogonality may be summarised by gathering equations (19)–(21) as follows

$$\begin{bmatrix} H(z) & H(-z) \\ G(z) & G(-z) \end{bmatrix} \begin{bmatrix} H(z^{-1}) & G(z^{-1)} \\ H(-z^{-1}) & G(-z^{-1}) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{22}$$

Filters that obey all of the conditions of orthogonality in addition to those of perfect reconstruction may be described as *canonical* filters. Perfect reconstruction is assured if, in addition to the conditions of (22), it is assumed that the synthesis filters are the time reversed versions of the analysis filters, such that $D(z) = G(z^{-1})$ and $E(z) = H(z^{-1})$.

Time-Domain Orthogonal Filters

A solution to the problem of achieving both perfect reconstruction and orthogonality with filters implemented in the time domain was provided by Smith and Barnwell (1984, 1986). They proposed to employ FIR filters, which have a finite number M of coefficients. This makes them eminently suited to online processing. However, such filters cannot obey the conditions of sequential orthogonality at two-point displacements and be symmetric about a central coefficient at the same time. Therefore, the filters, which must have an even number of coefficients, are bound to have a phase effect.

To see the necessity of an even number of coefficients, consider $G(z) = g_0 + g_1 z + \cdots + g_{M-1} z^{M-1}$ in the case where M is an odd number, as it must be if the coefficients are to be symmetric about a central point. Then, if $2n = M - 1$, the autocovariance at the corresponding even-valued lag is

$p_{2n} = g_0 g_{M-1} \neq 0$, since $g_0, g_{M-1} \neq 0$, by definition. Therefore, the condition of sequential orthogonality is violated.

In the process of the reconstruction of the input signal, the phase effects occasioned by asymmetric filters will be eliminated. Therefore, if signal reconstruction is the primary purpose, then such phase distortions can be ignored.

According to the prescription of Smith and Barnwell, the lowpass and highpass filters of the analysis section should bear the following relationship:

$$G(z) = z^{M-1}H(-z^{-1}) \qquad \text{and} \qquad H(z) = -z^{M-1}G(-z^{-1}). \tag{23}$$

To understand this construction, observe that, given a lowpass filter $G(z)$, a highpass filter would normally be derived by replacing z by $-z$ to give $H(z) = G(-z)$. When $z = \exp\{-\mathrm{i}\omega\}$, this amounts nothing more than the modulation of the filter coefficients by the factor $(-1)^j = \exp\{-\mathrm{i}\pi j\}$, where j is the coefficient index. The effect is the frequency shifting of the filter by π radians, which can also be described as a reflection of its frequency response about the vertical axis through $\pi/2$.

When $-z^{-1}$ is used instead of $-z$ in deriving the highpass filter, the accompanying multiplication by $-z^{M-1}$ ensures that, if the lowpass filter $G(z)$ is causal, then the highpass filter $H(z)$ will become a causal filter rather than an anti-causal filter.

Using the definitions of (23), and the orthogonality conditions of (19) and (20), the determinant of (14) is evaluated as

$$\begin{aligned}\Delta(z) &= G(z)H(-z) - H(z)G(-z)\\ &= z^{M-1}\left\{G(z)G(z^{-1}) + G(-z)G(-z^{-1})\right\} = 2z^{M-1}\\ &= z^{M-1}\left\{H(-z)H(-z^{-1}) + H(z)H(z^{-1})\right\} = 2z^{M-1}.\end{aligned} \tag{24}$$

According to equation (15), in order to ensure perfect reconstruction, the filters of the synthesis section should be

$$D(z) = \frac{2}{\Delta(z)}H(-z) = G(z^{-1}) \quad \text{and} \quad E(z) = \frac{-2}{\Delta(z)}G(-z) = H(z^{-1}), \tag{25}$$

where the definitions of (23) have provided the expressions for $H(-z)$ and $G(-z)$. Thus, the synthesis filters are the time-reversed versions of the corresponding analysis filters.

One should recognise that, if the analysis filters are causal, then the synthesis filters, as specified by (25), will be anti-causal. They will impose lags in reversed time, which are advances in other words, that are equal and opposite to those of the analysis filters. Therefore, the network will impose no delays overall. To render the synthesis filters causal, they should be multiplied by z^{M-1}, in which case, they would become $D(z) = H(-z)$ and $E(z) = -G(-z)$.

A further condition that is satisfied by the filters of (23) is that of lateral orthogonality:

$$G(z)H(z^{-1}) + G(-z)H(-z^{-1}) = 0. \qquad \textit{Lateral Orthogonality} \tag{26}$$

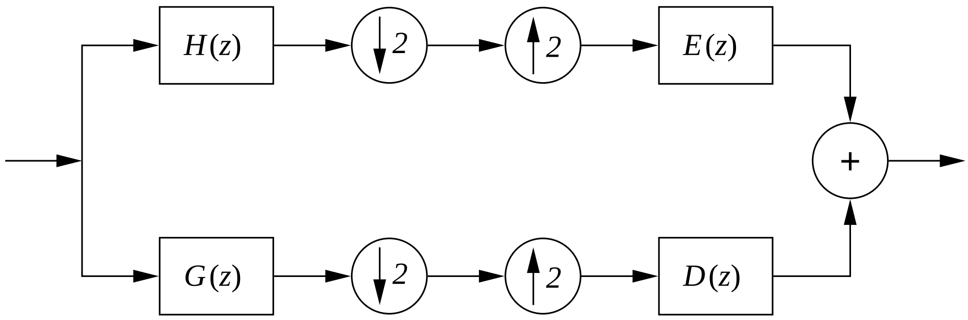

Figure 6. A depiction of the two-channel filter bank. If $H(z) = -z^{M-1}G(-z^{-1})$, then perfect reconstruction and orthogonality can be achieved by setting $E(z) = H(z^{-1})$ and $D(z) = G(z^{-1})$.

This is the alternated version of the cross covariance generating function of the two filter sequences. It can be confirmed in more than one way that the filters of (23) satisfy this condition. However, this follows easily from setting $G(z) = z^{M-1}H(-z^{-1})$ and from setting $G(-z) = -z^{M-1}H(z^{-1})$.

Next, we may confirm that the conditions of perfect reconstruction are indeed fulfilled, albeit that this is guaranteed by the specifications of (25). Substituting the expressions for $D(z)$ and for $E(z)$ into the equation $D(z)G(z)+E(z)H(z) = 2$ gives

$$D(z)G(z) + E(z)H(z) = \frac{2}{\Delta(z)}\{H(-z)G(z) - G(-z)H(z)\} = 2, \tag{27}$$

which provides the necessary confirmation of the condition of (8). On setting $D(z)G(z) = G(z^{-1})G(z)$ and $E(z)H(z) = H(z^{-1})H(z)$ in consequence of (23) and (25), equation (27) becomes

$$H(z)H(z^{-1}) + G(z)G(z^{-1}) = 2. \quad \textit{Power Complementarity} \tag{28}$$

This condition, which indicates that the superimposition of the squared gains of the two filters is a constant function, is described as the power complementarity of the filters.

Next to be confirmed is the condition $D(z)G(-z)+E(z)H(-z) = 0$, which is necessary for the cancellation of the aliasing effect. To show that this is satisfied, we substitute the expressions for $D(z)$ and $E(z)$ of (25) into the equation to show that

$$D(z)G(-z) + E(z)H(-z) = \frac{2}{\Delta(z)}\{H(-z)G(-z) - G(-z)H(-z)\} = 0. \tag{29}$$

Orthogonal Filters in the Frequency Domain

Although the FIR filters that are employed in two-channel filter banks are bound to be asymmetric, it is possible, in theory, to achieve perfect reconstruction and orthogonality with symmetric IIR filters that have an infinite number of coefficients. Such filters are bound to have finite supports in the frequency domain. Therefore, it is appropriate to implement them in the frequency domain. The filters cannot be employed in real-time signal processing and, for that reason, they have been largely ignored in the engineering literature. Nevertheless, they are of interest to statisticians working off line with recorded data.

Let us assume that the lowpass filter $G(z) = G(z^{-1})$ is a symmetric half-band filter, and that $G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2$, which is the condition of lowpass sequential orthogonality. We may specify that

$$G(z) = zH(-z) \quad \text{and} \quad G(-z) = -zH(z) \tag{30}$$

and that

$$H(-z) = z^{-1}G(z) \quad \text{and} \quad H(z) = -z^{-1}G(-z). \tag{31}$$

The condition of power complementarity is clearly satisfied.

The synthesis filters will be given by

$$D(z) = G(z) = G(z^{-1}) \quad \text{and} \quad E(z) = -zG(-z) = H(z^{-1}). \tag{32}$$

Thus, the lowpass synthesis filter is the same as the lowpass analysis filter. The one-point displacement associated with the highpass filter in the analysis section is followed by a one-point displacement in the opposite direction within the synthesis section.

It is straightforward to confirm that the conditions for perfect reconstruction are fulfilled. Thus, equation (9) for the cancellation of the aliasing effect is satisfied, since

$$D(z)G(-z) + E(z)H(-z) = G(z)G(-z) + \{-zG(-z)\}\{z^{-1}G(z)\} = 0 \tag{33}$$

Also, equation (8) becomes

$$\begin{aligned} D(z)G(z) + E(z)H(z) &= G(z)G(z) + \{-zG(-z)\}\{-z^{-1}G(-z)\} \\ &= G(z)G(z) + G(-z)G(-z) \\ &= G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2. \end{aligned} \tag{34}$$

The last of these equations expresses the sequential orthogonality of the lowpass channel, which is already one of the assumptions. Using the expressions for $G(z)$ and $G(-z)$ from (30) within the final expression of (34), and by setting $G(z^{-1}) = z^{-1}H(-z^{-1})$ and $G(-z^{-1}) = -z^{-1}H(z^{-1})$, we get

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2. \tag{35}$$

This denotes the sequential orthogonality of the highpass channel. Finally, we observe that

$$G(z)H(z) + G(-z)H(-z) = G(z)H(z) + \{-zH(z)\}\{z^{-1}G(z)\} = 0, \tag{36}$$

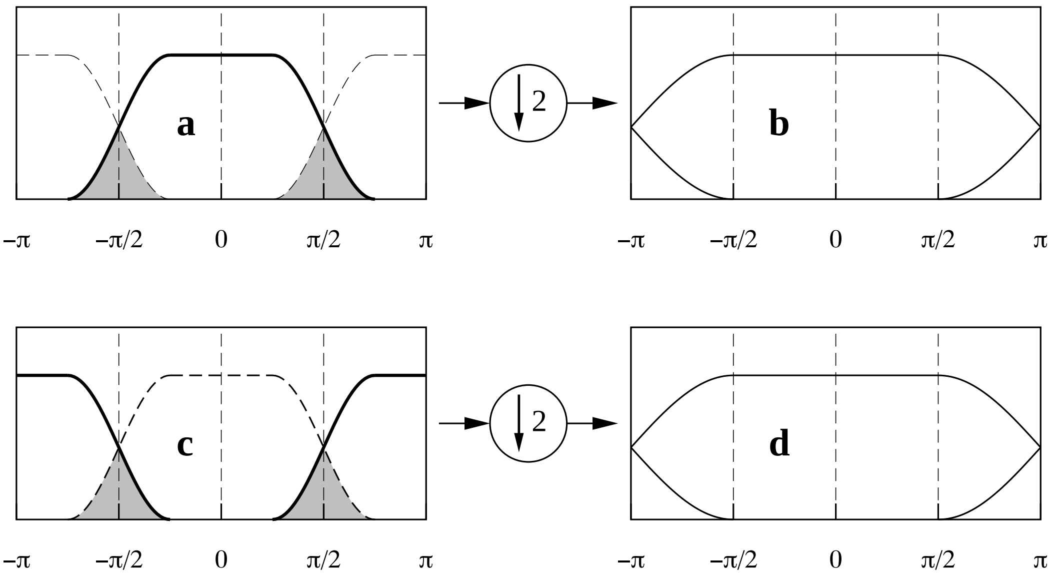

Figure 7. The effects downsampling on the squared gain of the lowpass filter $G(z)$ (top) and on the lowpass filter $H(z)$ (bottom).

which denotes the lateral orthogonality of the two channels.

The Gain of the Canonical Filters

The canonical filters that obey the conditions of orthogonality as well as the conditions of perfect reconstruction have characteristics that can be shown to advantage in some simple diagrams relating to the squared gain of the filters.

In Figure 7, in the diagram labelled (a), the envelope centred on the zero frequency and bounded by a continuous line represents the squared gain of a half-band lowpass filter $G(z)$, which is $G(z)G(z^{-1})$. The complementary highpass filter $H(z)$ has a squared gain $H(z)H(z^{-1})$ that is represented in diagram (a) by a broken line and in diagram (c) by a continuous line. On the RHS of the diagrams are the squared gains of the filters, as they would be in the case of downsampled data.

On the LHS of Figure 7, in diagrams (a) and (c), the transitions of the two filters between their stop bands and their pass bands occur in the vicinities of $-\pi/2$ and $\pi/2$. There, the profile of the squared gain of the lowpass filter is the mirror image of that of the highpass filter. The sum of the two is a constant function with a value of 2. The equation that represents this sum is $G(z)G(z^{-1}) + H(z)H(z^{-1}) = 2$, which is the condition of power complementarity.

Since either $H(z) = -z^{M-1}G(-z^{-1})$, in the case of the canonical time-domain filters or $H(z) = -zG(-z^{-1})$ with $G(z^{-1}) = G(z)$, in case of the canonical frequency-domain fiters, the equation of power complementarity can be rendered as $G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 2$, which expresses the conditions for the sequential orthogonality of the filters. These conditions represents the effects on the autocovariance generating functions of replacing the coefficients with odd-valued indices by with zeros.

The effect of downsampling is to dilate the squared gain functions. The function that was formerly supported on the interval $[-\pi, \pi]$ of length 2π is now supported on an interval of length 4π, which will be wrapped twice around a circle of circumference 2π. The effect upon the squared gain of the lowpass filter is represented in the diagram labelled (b), whereas the effect of the downsampling upon the highpass filter is represented by the diagram labelled (d).

The diagrams (b) and (d), which appear to be identical, arise in different ways. In the transition from (a) to (b) the processes of downsampling and wrapping cause mirror-image reversals of the shaded regions of (a) that lie in the sub intervals $[-\pi, -\pi/2]$ and $[\pi/2, \pi]$ and which correspond to minor parts of the low-frequency pass band that extend beyond its nominal limits. The reversed images are carried across the origin to the opposite sides of the interval $[-\pi.\pi]$.

In the transition from (c) to (d) the same effects of image reversal and translation are applied to the major parts of the high-freqency passband that lie within the same sub intervals $[-\pi, -\pi/2]$ and $[\pi/2, \pi]$. The reversed and dilated images are mapped into the intervals $[0, \pi]$ and $[-\pi, 0]$ respectively.

The process of cancelling the aliasing effects can be understood in terms of these diagrams. The aliasing gives rise to the shaded regions of the diagrams on the LHS. In diagram (a), the effects are due to the filter product $D(z)G(-z) = G(z^{-1})G(-z)$. In diagram (c), they are due to the product $E(z)H(-z) = H(z^{-1})H(-z) = -G(z^{-1})G(-z)$. Evidently, these two effects will cancel.

The diagrams of Figure 7 refer to the squared gains of the filters. This conceals the fact that the aliasing effects of the two filters, which are represented by the shaded regions, are of opposite signs and that they cancel each other.

Processing in Two Phases

From the point of view of real-time signal processing, the scheme that is represented by equations (1) and (2) is more time consuming and more demanding of storage or memory than it need be. Given that only half of the processed data is transmitted after the downsampling, it is clear that twice the necessary operations have been entailed in the filtering processes. This redundancy can be overcome by splitting the data into the sequences indexed by the odd and the even numbers.

Given that

$$\begin{aligned} y(z) &= \cdots + y_0 + y_1 z + y_2 z^2 + y_3 z^3 + y_4 z^4 + \cdots, \\ y(-z) &= \cdots + y_0 - y_1 z + y_2 z^2 - y_3 z^3 + y_4 z^4 - \cdots, \end{aligned} \tag{37}$$

the z-transforms of the even and odd sequences are available as follows:

$$\begin{aligned} y^e(z^2) &= \frac{1}{2}\{y(z) + y(-z)\} = \cdots + y_0 + y_2 z^2 + y_4 z^4 + y_6 z^6 + \cdots, \\ zy^o(z^2) &= \frac{1}{2}\{y(z) - y(-z)\} = \cdots + y_1 z + y_3 z^3 + y_5 z^5 + y_5 z^7 + \cdots; \end{aligned} \tag{38}$$

and it can be seen that $y^e(z)$ is just the z-transform of the downsampled sequence $y(t \downarrow 2)$. To obtain $y^o(z^2)$, the second of these should be multiplied by z^{-1}.

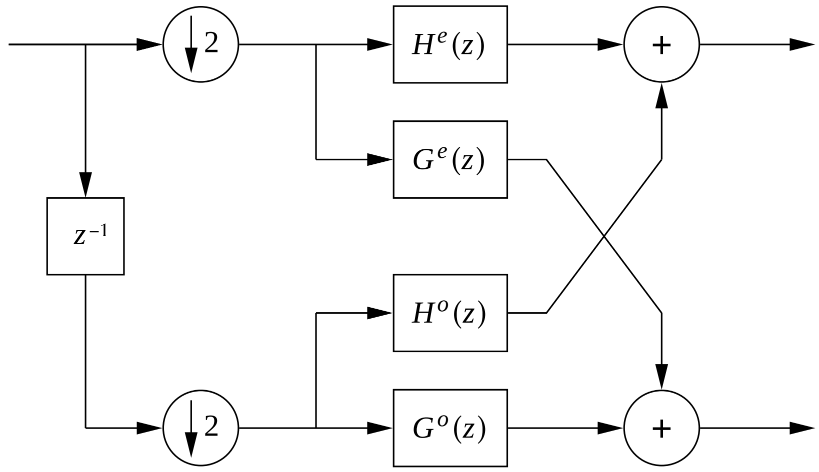

Figure 8. The analysis stage of the two-channel filter bank, which separates the data points bearing even indices from those bearing odd indices.

Observe that the presence of the argument z^2 within $y^e(z^2)$ and $y^o(z^2)$ implies that there are alternate zeros within the respective data sequences. Replacing z^2 by z is tantamount to eliminating these zeros. The z-transform of the original sequence can be recovered by adding the two series of (38):

$$y(z) = y^e(z^2) + zy^o(z^2). \tag{39}$$

The transfer functions of the filters can be partitioned in an analogous manner. Thus

$$\begin{aligned} G^e(z^2) &= \frac{1}{2}\{G(z) + G(-z)\}, \\ zG^o(z^2) &= \frac{1}{2}\{G(z) - G(-z)\} \end{aligned} \tag{40}$$

are the components of the lowpass filter

$$G(z) = G^e(z^2) + zG^o(z^2). \tag{41}$$

The highpass filter can be expressed likewise in terms of its components:

$$H(z) = H^e(z^2) + zH^o(z^2). \tag{42}$$

The relations between the two filters and their odd and even components can be expressed via the following equations:

$$\begin{bmatrix} H^e(z^2) & H^o(z^2) \\ G^e(z^2) & G^o(z^2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H(z) & H(-z) \\ G(z) & G(-z) \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}. \tag{43}$$

The matrix on the LHS can be described as the analysis matrix of the two-phase processing. The matrix in the middle of the RHS is that of a two-point discrete Fourier transform.

The even-indexed lowpass signal that emerges from downsampling the transform $G(z)y(z)$ is

$$\begin{aligned}\gamma(z^2) = \frac{1}{2}\{\gamma(z)+\gamma(-z)\} &= \frac{1}{2}\{G(z)y(z)+G(-z)y(-z)\},\\ &= \frac{1}{4}\{G(z)+G(-z)\}\{y(z)+y(-z)\},\\ &\quad + \frac{1}{4}\{G(z)-G(-z)\}\{y(z)-y(-z)\},\\ &= G^e(z^2)y^e(z^2)+z^2G^o(z^2)y^o(z^2).\end{aligned} \tag{44}$$

This signal combines both the odd and the even indexed subsequences of the data. Therefore, in Figure 8, which describes the two-phase two-channel network, there is a cross-over between the two channels. Prior to downsampling, an advance is created in the odd channel by the operator z^{-1}. This brings the odd-indexed elements into the places formerly occupied by the even-indexed elements, which are then discarded.

The analogous highpass signal is

$$\beta(z^2) = H^e(z^2)y^e(z^2)+z^2H^o(z^2)y^o(z^2). \tag{45}$$

Combining (44) and (45) gives

$$\begin{aligned}\frac{1}{2}\begin{bmatrix} H(z) & H(-z)\\ G(z) & G(-z)\end{bmatrix}\begin{bmatrix} y(z)\\ y(-z)\end{bmatrix} &= \begin{bmatrix} H^e(z^2) & z^2H^o(z^2)\\ G^e(z^2) & z^2G^o(z^2)\end{bmatrix}\begin{bmatrix} y^e(z^2)\\ y^o(z^2)\end{bmatrix}\\ &= \begin{bmatrix}\beta(z^2)\\ \gamma(z^2)\end{bmatrix},\end{aligned} \tag{46}$$

The expression on the LHS represents the effect that the subsequent downsampling and upsampling have on the direct transformations $H(z)y(z)$ and $G(z)y(z)$. The second expression represents the transformation that occurs after the separation of the two phases of the data. The effects, represented by the vector on the RHS, are the same whichever process is adopted.

The arguments of $\beta(z^2)$ and $\beta(z^2)$ can be taken to imply the presence of alternate zero elements within the signal sequences, which may be construed as the effect of the subsequent upsampling. It will be observed that, within Figure 8, the arguments of the filters are z rather than z^2, as they are in (46). This is because, in this structure, the downsampling operations precede the filtering.

We seek the specification of the two-phase synthesis filters that will ensure the orthogonality of the outputs of the two channels as well as the perfect reconstruction of the input. Perfect reconstruction will be achieved if and only if the synthesis matrix of the two-phase network, comprising the filters of both channels, is the inverse of the corresponding analysis matrix of (43).

It may be assumed that filters of the direct form are subject to the orthogonality conditions of (22), which are

$$\frac{1}{2}\begin{bmatrix} H(z) & H(-z)\\ G(z) & G(-z)\end{bmatrix}\begin{bmatrix} H(z^{-1}) & G(z^{-1})\\ H(-z^{-1}) & G(-z^{-1})\end{bmatrix} = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}, \tag{47}$$

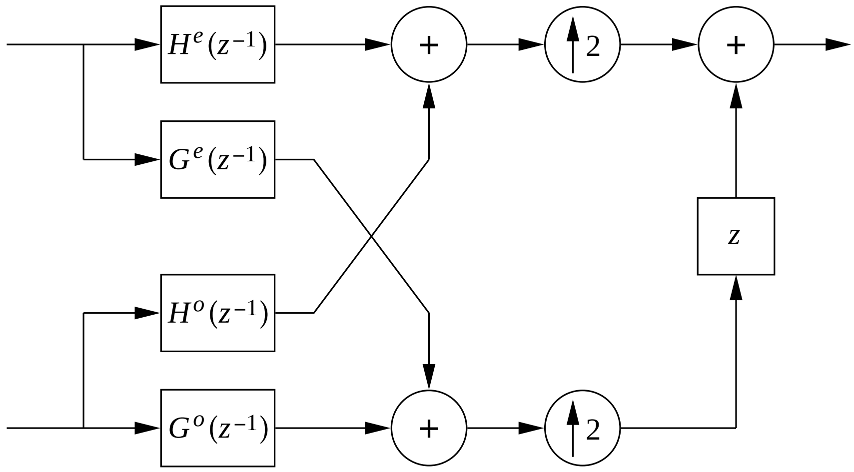

Figure 98. The synthesis stage of the two-channel filter bank, which separates the data points bearing even indices from those bearing odd indices.

Then, it is straightforward to show that the inverse of the matrix of (43) is just its conjugate transpose:

$$\begin{bmatrix} H^e(z^{-2}) & G^e(z^{-2}) \\ H^o(z^{-2}) & G^o(z^{-2}) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} H(z^{-1}) & G(z^{-1}) \\ H(-z^{-1}) & G(-z^{-1}) \end{bmatrix}. \tag{48}$$

The first two matrices on the RHS of (48) cancel with the last two matrices on the RHS of (43), leaving an identity matrix times the factor of 1/2. The remaining product is that of (47). The upshot is is that

$$\begin{bmatrix} H^e(z) & H^o(z) \\ G^e(z) & G^o(z) \end{bmatrix}\begin{bmatrix} H^e(z^{-1}) & G^e(z^{-1}) \\ H^o(z^{-1}) & G^o(z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{49}$$

from which the lateral orthogonality of the filters is manifest.Their sequential orthogonality is already guaranteed by (47). The matrices of equation (49) are unitary only when $z = \exp\{-\mathrm{i}\omega\}$ is a point on the unit circle. Therefore, they are described as *paraunitary*—meaning distinct from but analogous to unitary matrices. A synonym for paraunitary is *lossless.*

In terms of the z-transforms, the output of the synthesis transform is

$$\begin{aligned} y^e(z) &= H^e(z^{-2})\beta(z^2) + G^e(z^{-2})\gamma(z^2), \\ y^o(z) &= H^o(z^{-2})\beta(z^2) + G^o(z^{-2})\gamma(z^2). \end{aligned} \tag{50}$$

These can be recombined to give

$$y(z) = y^e(z^2) + zy^o(z^2). \tag{51}$$

In Figure 9, which describes the synthesis stage of the two-phase two-channel network, a delay is imposed on the odd channel by the operator z, This is a counterpart to the advance that has been created in the analytic stage prior to downsampling, which is depicted in figure 8.

The canonical conditions of perfect reconstruction and orthogonality, which are assumed to apply to the two-phase network, continue to allow some latitude in the specification of the filters. However, conditions that apply to the canonical filters of the direct form will be inherited by the two-phase filters. In particular, if these are of the FIR variety, then the relationship of (23) between $H(z)$ and $G(z)$ will be inherited by the two-phase filters. To see the implications, consider the following two expressions:

$$H^e(z^2) = \frac{1}{2}\{H(z) + H(-z)\} \quad \text{and} \quad zH^o(z^2) = \frac{1}{2}\{H(z) - H(-z)\}. \tag{52}$$

Into these, one must substitute

$$H(z) = -z^{M-1}G(-z^{-1}) \quad \text{and} \quad H(-z) = z^{M-1}G(z^{-1}), \tag{53}$$

which come from (23). The outcomes are

$$H^e(z^2) = z^{M-2}G^o(z^{-2}) \quad \text{and} \quad z^2H^o(z^2) = -z^MG^e(z^2), \tag{54}$$

A Lattice Structure for the Two-Phase Network

An efficient way of implementing a paraunitary filter bank within a two-channel network is to employ a lattice structure in which successive segments consist of a product of an elementary Givens matrix R, which is an othonormal matrix that effects a planar rotation, and a delay matrix $\Lambda(z)$, which is paraunitary.

We define

$$E(z) = \begin{bmatrix} H^e(z^2) & H^o(z^2) \\ G^e(z^2) & G^o(z^2) \end{bmatrix}, \tag{55}$$

which is the matrix of the analysis section. This can be factorised as

$$E(z) = R_N\Lambda(z)R_{N-1}\Lambda(z)\cdots R_1\Lambda(z)R_0 \tag{56}$$

with

$$R_k = \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \quad \text{and} \quad \Lambda(z) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}. \tag{57}$$

Using $\alpha = \tan\theta$, the kth Givens matrix can be written as

$$R_k = \cos\theta_k \begin{bmatrix} 1 & \alpha_k \\ -\alpha_k & 1 \end{bmatrix} = \cos\theta_k Q_k. \tag{58}$$

There is $\cos\theta = 1/\sqrt{1+\tan^2\theta}$, and, on defining $\gamma = \prod_{k=0}^{n}(1/\sqrt{1+\alpha_k})$, the product of (56) can be cast in the form of

$$E(z) = \gamma Q_N\Lambda(z)Q_{N-1}\Lambda(z)\cdots Q_1\Lambda(z)Q_0. \tag{59}$$

This enables a significant reduction in the number of evaluations of trigonometrical functions, which are computationally expensive to perform. The matrix of

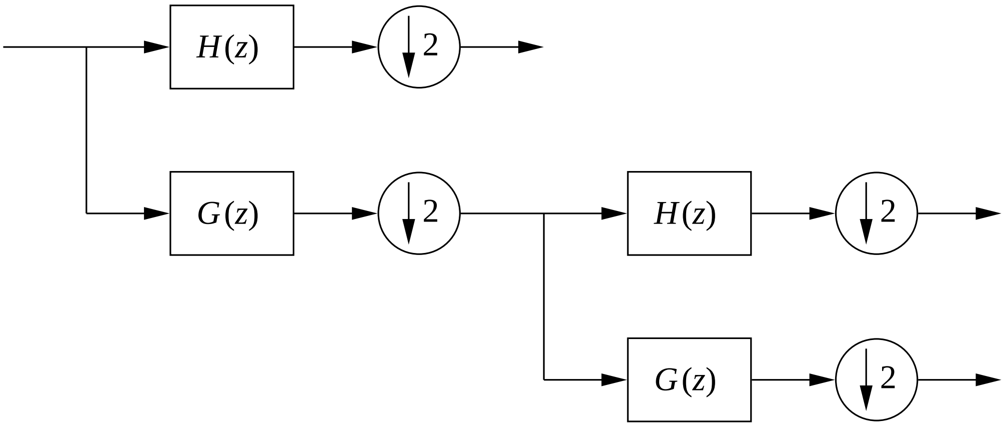

Figure 10. The analysis section of a dyadic filter bank, expressed in terms of z-transform polynomials.

the synthesis stage is just the conjugate transpose of the matrix of the analysis stage.

Given the analysis filters of the direct-form, which are

$$\begin{bmatrix} H(z) \\ G(z) \end{bmatrix} = \begin{bmatrix} H^e(z^2) & H^o(z^2) \\ G^e(z^2) & G^o(z^2) \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix}, \tag{60}$$

the segments of the lattice can be derived by a simple recursion. The analysis filters of the kth stage are derived from those of the preceding stage via

$$\begin{bmatrix} H^{(k)}(z) \\ G^{(k)}(z) \end{bmatrix} = \begin{bmatrix} 1 & z\alpha_k \\ -\alpha_k & z \end{bmatrix} \begin{bmatrix} H^{(k-1)}(z) \\ G^{(k-1)}(z) \end{bmatrix}. \tag{61}$$

The inverse relationship is

$$z(1+\alpha_k^2) \begin{bmatrix} H^{(k-1)}(z) \\ G^{(k-1)}(z) \end{bmatrix} = \begin{bmatrix} z & -z\alpha_k \\ \alpha_k & 1 \end{bmatrix} \begin{bmatrix} H^{(k)}(z) \\ G^{(k)}(z) \end{bmatrix}. \tag{62}$$

In order for the degrees of the polynomials on the LHS to be one less than the degrees of those on the RHS, a value of α_k must be found that will ensure that the highest power of z in $H^{(k)}(z) - \alpha G^{(k)}(z)$ is eliminated.

Successive Stages of a Dyadic Decomposition

The analysis section of the two-channel filter bank, which entails filtering and downsampling, can be replicated successively within the lowpass channel to effect a dyadic decomposition of the Nyquist frequency range, which runs from zero to π radians per sample interval.

Thus, a succession of frequency intervals are generated in the form of

$$(\pi/2^j, \pi/2^{j-1}); j = 1, 2, ..., n \tag{63}$$

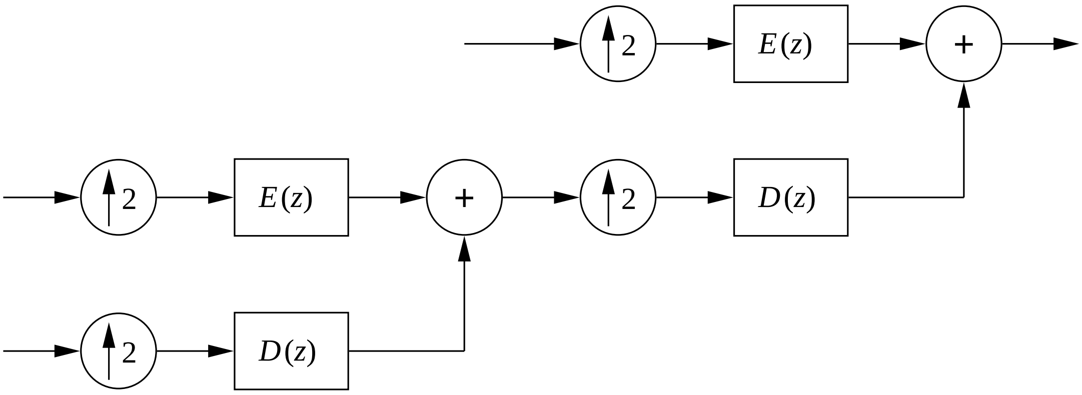

Figure 11. The synthesis section of a dyadic filter bank, expressed in terms of z-transform polynomials.

of which the bandwidths are halved repeatedly in the descent from high frequencies to low frequencies. In the case of a finite data sequence of T points, the sequence of bandwidths must end when $T/2^j$ is no longer divisible by 2.

To illustrate the analysis stage of a dyadic decomposition of the frequency range, it is enough to consider a filter bank with two stages of decomposition. This is illustrated in Figure 10. Successive stages can be added in a self-evident manner. If the conditions of sequential and lateral orthogonality prevail within the two-channel network, then they will prevail within and between the channels of the derived network.

The result may be described as a logarithmic filter bank, since the logarithm of the dyadic scale contains horizontal bands equal width. It may also be described as an octave-band filter bank, since successive highpass outputs contain an octave of the bandwidth of the input signal

The synthesis stage of a dyadic decomposition involves a straightforward reversal of the analysis stage. The reconstruction of the input signal begins at the lowest frequency level of the decomposition at which a two-channel split has occurred.

By completing the synthesis stage at that level, the split is mended and, if the conditions perfect reconstruction prevail, then the result will be as if the split had not occurred. Then the split at the next lowest level is mended, and so on upwards, until the input signal has been reconstructed. The synthesis stage to accompany the analysis stage of Figure 109 is illustrated in Figure 11.

A Decomposition of Equal Bandwidths

A dyadic decomposition that proceeds through n iterations, by successively dividing the lowest frequency band, generates $n + 1$ octave bands. However, it is possible to subdivide both the low-frequency and the high frequency band; and, in n successive iterations, it is possible to generate 2^n bands of equal width. In that case, care must be take to ensure that these frequency bands are arranged in an appropriate order.

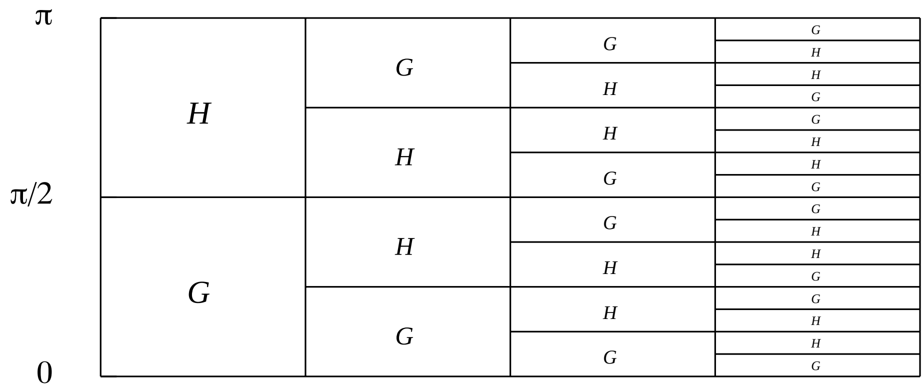

Figure 12. The succession of filters that are employed in dividing the frequency range $[0, \pi]$ into 16 bands of equal width. The lowpass and highpass filters, which are followed by downsampling operations, are indicated by the letters G and H, respectively.

The upper tranche of Figure 4 shows the effect of the high pass filter $H(z)$ and of the succeeding downsampling operation ($\downarrow 2$) on the spectrum of the data. A dilated spectrum is derived that appears to be in a reversed order of frequency.

The original positive and negative high-frequency spectral components of the data, which are supported on the half intervals $[\pi, \pi/2]$ and $[-\pi/2, -\pi]$, and which are mirror images, are dilated through downsampling, and then they are mapped crosswise onto the intervals $[-\pi, 0]$ and $[0, \pi]$, respectively. In fact, this reversal is the effect of the the circular wrapping of the the dilated spectrum; and the situation would be amended by applying a rotation of π radians or $180°$ degrees to the derived spectrum so as to interchange 0 and $\pm\pi$.

If it is intended, subsequently, to isolate the components within the unamended spectrum of the data that are supported on the intervals $[-\pi, -3\pi/4]$ and $[3\pi/4, \pi]$, which comprise the highest frequencies, then a lowpass filter $G(z)$ must be applied to these reversed components, followed by the downsampling operation. If $H(z)$ is applied instead of $G(z)$, then the low-frequency components will be isolated, and the subsequent downsampling will correct the reversal.

In order to generate a set of frequency bands in the natural order of their frequency ranges, one must take account of these reversals. Figure 12 indicates how this should be done. However, although the bands are in the natural order, the contents of high-frequency bands within alternate stages of the decomposition will be in the reversed order of frequency.

Apart from the problems of image reversals, the recursive tree structure is inefficient in its implementation, since the successive filters are liable to entail numerous numerical operations. Simpler parallel structures, that have a single filter in each channel, have been by derived by Esteban and Galand (1977) and by Cheung and Winslow (1980). The number of the equal bandwidth channels is restricted to be a power of 2. Less restrictive structures that have an arbitrary number of channels of equal bandwidth will be presented in Chapter 8.

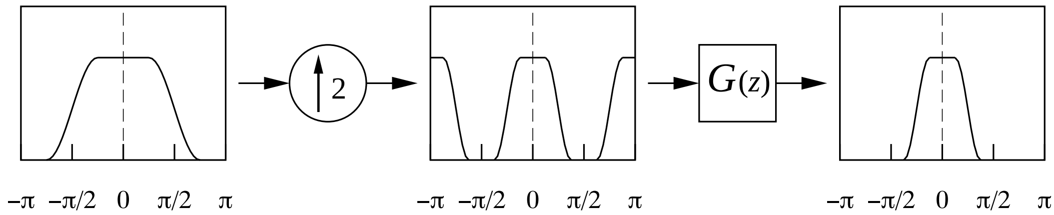

Figure 13. The formation of the quarter-band lowpass filter $G(z)G(z^2)$, where $G(z)$ is the half-band lowpass filter and where $G(z^2)$ is the upsampled version.

Filtering without Downsampling

As the process of dyadic filtering proceeds from one stage to the next, the bandwidth of the resulting signal is halved. The downsampling that accompanies the process ensures that information content of the signal is conveyed by a minimum number of elements. Thus, the output of the second stage of the dyadic decomposition comprises half the number of elements of the output of the first stage.

Often, it is desired to make direct comparisons between the signals emerging from different stages of the decomposition. For this purpose, it is necessary to dispense with downsampling. Then, the resulting sequences can be compared, element for element, by their juxtaposition. This cannot be achieved by using the same lowpass and highpass filters at every stage of the decomposition. The filters must the adapted to each stage of the decomposition.

The filter sequence that generates the low-frequency signal in the nth stage of the decimated dyadic decomposition is

$$G(z)(\downarrow 2)G(z)(\downarrow 2)\cdots G(z)(\downarrow 2,) \tag{65}$$

which comprises n identical filters, each followed by a downsampling operation. This must be replaced by

$$G(z)G(z^2)\cdots G(z^{2n}.) \tag{66}$$

It is necessary to explain how this generates the appropriate lowpass filter. For this, it is enough to show how the half-band lowpass filter $G(z)$ is succeeded by the quater-band lowpass filter $G(z)G(z^2)$. This is illustrated in Figure 13.

To begin, we may note that replacing z by z^2, which corresponds to the process of upsampling, multiplies the frequency argument ω by 2 to create two cycles in place of one as ω traverses the interval from $-\pi$ to π. The imaging effect on the frequency response of the lowpass filter is show in the second box of Figure 13.

The third box of the figure shows the effect of compounding $G(z^2)$ with the half-band lowpass filter $G(z)$. The intended effect is to preserve the spectrum over the sub interval $[-\pi/2, \pi/2]$ and to nullify it over remainder of the interval, thereby generating the quarter-band lowpass filter.

The procedure that avoids dowsampling and that generates full-length vectors at each stage of the dyadic decomposition is described as a Maximum-Overlap Discrete Wavelets Transform (MODWT).

References

Cheung, R.S., and R.L. Winslow, (1980), High Quality 16 KBS Voice Transmission: The Subband Coder Approach, *IEEE International Conference on Acoustics, Speech and Signal Processing: ICASSP 80,* 319–322

Croisier, A., D. Esteban, and C. Galand, (1976), Perfect Channel Splitting by Use of Interpolation, Decimation, Tree Decomposition Techniques, *Proceedings of the International Conference on Information Sciences and Systems,* 443–446, Patras, Greeece, August 1976

Esteban, D., and C. Galand, (1977), Application of Quadrature Mirror Filters to Split Band Voice Coding Schemes. *IEEE International Conference on Acoustics Speech, and Signal Processing: ICASSP 77,* 191–195.

Esteban D., and C. Garland, (1983), Design and Evaluation of Parallel Approach to Quadrature Mirror Filters, *IEEE International Conference on Acoustics, Speech and Signal Processing: ICASSP 83,* 224–227.

Johnston, J., (1980), A Filter Family Designed for Use in Quadrature Mirror Filter Banks, *IEEE International Conference on Acoustics, Speech and Signal Processing: ICASSP 80*, 291–294.

Smith M.J.T., and T.P. Barnwell, (1984), A Procedure for Designing Exact Reconstruction Filter Banks for Tree-structured Subband Coders, *IEEE International Conference on Acoustics, Speech and Signal Processing: ICASSP 84,* 421–424.

Smith M.J.T., and T.P. Barnwell III, (1986), Exact Reconstruction for Tree-structured Subband Coders, *IEEE Transactions on Acoustics, Speech, and Signal Processing,* 34, 434–441.