LECTURE 4

Time-Series Analysis in the Frequency Domain

A sequence is a function mapping from a set of integers, described as the index set, onto the real line or into a subset thereof. A time series is a sequence whose index corresponds to consecutive dates separated by a unit time interval.

In the statistical analysis of time series, the elements of the sequence are regarded as a set of random variables. Usually, no notational distinction is made between these random variables and their realised values. It is important nevertheless to bear the distinction in mind.

In order to analyse a statistical time series, it must be assumed that the structure of the statistical or stochastic process which generates the observations is essentially invariant through time. The conventional assumptions are summarised in the condition of stationarity. In its strong form, the condition requires that any two segments of equal length which are extracted from the time series must have identical multivariate probability density functions. The condition of weak stationarity requires only that the elements of the time series should have a common finite expected value and that the autocovariance of two elements should depend only on their temporal separation.

A fundamental process, from which many other stationary processes may be derived, is the so-called white-noise process which consists of a sequence of uncorrelated random variables, each with a zero mean and the same finite variance. By passing white noise through a linear filter, a sequence whose elements are serially correlated can be generated. In fact, virtually every stationary stochastic process may be depicted as the product of a filtering operation applied to white noise. This result follows from the Cramér–Wold Theorem which will be presented after we have introduced the concepts underlying the spectral representation of a time series.

The spectral representation is rooted in the basic notion of Fourier analysis which is that well-behaved functions can be approximated over a finite interval, to any degree of accuracy, by a weighted combination of sine and cosine functions whose harmonically rising frequencies are integral multiples of a fundamental frequency. Such linear combinations are described as Fourier sums or Fourier series. Of course, the notion applies to sequences as well; for any

number of well-behaved functions may be interpolated through the coordinates of a finite sequence.

We shall approach the Fourier analysis of stochastic processes via the exact Fourier representation of a finite sequence. This is extended to provide a representation of an infinite sequence in terms of an infinity of trigonometrical functions whose frequencies range continuously in the interval $[0, \pi]$. The trigonometrical functions and their weighting functions are gathered under a Fourier–Stieltjes integral. It is remarkable that, whereas a Fourier sum serves only to define a strictly periodic function, a Fourier integral suffices to represent an aperiodic time series generated by a stationary stochastic process.

The Fourier integral is also used to represent the underlying stochastic process. This is achieved by describing the stochastic processes which generate the weighting functions. There are two such weighting processes, associated respectively with the sine and cosine functions; and their common variance, which is a function $f(\omega)$, $\omega \in [0, \pi]$, is the so-called spectral density function.

The relationship between the spectral density function and the sequence of autocovariances, which is summarised in the Wiener–Khintchine theorem, provides a link between the time-domain and the frequency-domain analyses. The sequence of autocovariances may be obtained from the Fourier transform of the spectral density function and the spectral density function is, conversely, a Fourier transform of the autocovariances.

Stationarity

Consider two vectors of $n + 1$ consecutive elements from the process $y(t)$:

$$(4.1) \qquad [y_t, y_{t+1}, \ldots, y_{t+n}] \qquad \text{and} \qquad [y_s, y_{s+1}, \ldots, y_{s+n}]$$

Then $y(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ is strictly stationary if the joint probability density functions of the two vectors are the same for any values of t and s regardless of the size of n. On the assumption that the first and second-order moments of the distribution are finite, the condition of stationarity implies that all the elements of $y(t)$ have the same expected value and that the covariance between any pair of elements of the sequences is a function only of their temporal separation. Thus,

$$(4.2) \qquad E(y_t) = \mu \qquad \text{and} \qquad C(y_t, y_s) = \gamma_{|t-s|}.$$

On their own, the conditions of (2) constitute the conditions of weak stationarity.

A normal process is completely characterised by its mean and its autocovariances. Therefore, a normal process $y(t)$ which satisfies the conditions for weak stationarity is also stationary in the strict sense.

The Autocovariance Function

The covariance between two elements y_t and y_s of a process $y(t)$ which are separated by $\tau - |t - s|$ intervals of time, is known as the autocovariance at lag τ and is denoted by γ_τ. The autocorrelation at lag τ, denoted by ρ_τ, is defined by

$$(4.3) \qquad \rho_\tau = \frac{\gamma_\tau}{\gamma_0,}$$

where γ_0 is the variance of the process $y(t)$.

The stationarity conditions imply that the autocovariances of $y(t)$ satisfy the equality

$$(4.4) \qquad \gamma_\tau = \gamma_{-\tau}$$

for all values of τ.

The autocovariance matrix of a stationary process corresponding to the n elements $y_0, y_1, \ldots, y_{n-1}$ is given by

$$(4.5) \qquad \Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{n-2} \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_0 \end{bmatrix}.$$

The sequences $\{\gamma_\tau\}$ and $\{\rho_\tau\}$ are described as the autocovariance and autocorrelation functions respectively.

The Filtering of White Noise

A white-noise process is a sequence $\varepsilon(t)$ of uncorrelated random variables with mean zero and common variance σ_ε^2. Thus

$$(4.6) \qquad \begin{aligned} E(\varepsilon_t) &= 0, \quad \text{for all } t \\ E(\varepsilon_t \varepsilon_s) &= \begin{cases} \sigma_\varepsilon^2, & \text{if } t = s; \\ 0, & \text{if } t \neq s. \end{cases} \end{aligned}$$

By a process of linear filtering, a variety of time series may be constructed whose elements display complex interdependencies. A finite linear filter, also called a moving-average operator, is a polynomial in the lag operator of the form $\mu(L) = \mu_0 + \mu_1 L + \cdots + \mu_q L^q$. The effect of this filter on $\varepsilon(t)$ is described by the equation

$$(4.7) \qquad \begin{aligned} y(t) &= \mu(L)\varepsilon(t) \\ &= \mu_0 \varepsilon(t) + \mu_1 \varepsilon(t-1) + \mu_2 \varepsilon(t-2) + \cdots + \mu_q \varepsilon(t-q) \\ &= \sum_{i=0}^{q} \mu_i \varepsilon(t-i). \end{aligned}$$

The operator $\mu(L)$ is also be described as the transfer function which maps the input sequence $\varepsilon(t)$ into the output sequence $y(t)$.

An operator $\mu(L) = \{\mu_0 + \mu_1 L + \mu_2 L^2 + \cdots\}$ with an indefinite number of terms in rising powers of L may also be considered. However, for this to be practical, the coefficients $\{\mu_0, \mu_1, \mu_2, \ldots\}$ must be functions of a limited number of fundamental parameters. In addition, it is required that

$$(4.8) \qquad\qquad\qquad \sum_i |\mu_i| < \infty.$$

Given the value of $\sigma_\varepsilon^2 = V\{\varepsilon(t)\}$, the autocovariances of the filtered sequence $y(t) = \mu(L)\varepsilon(t)$ may be determined by evaluating the expression

$$
\begin{aligned}
(4.9) \qquad \gamma_\tau &= E(y_t y_{t-\tau}) \\
&= E\left(\sum_i \mu_i \varepsilon_{t-i} \sum_j \mu_j \varepsilon_{t-\tau-j} \right) \\
&= \sum_i \sum_j \mu_i \mu_j E(\varepsilon_{t-i} \varepsilon_{t-\tau-j}).
\end{aligned}
$$

From equation (6), it follows that

$$(4.10) \qquad\qquad\qquad \gamma_\tau = \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau};$$

and so the variance of the filtered sequence is

$$(4.11) \qquad\qquad\qquad \gamma_0 = \sigma_\varepsilon^2 \sum_j \mu_j^2.$$

The condition under equation (8) guarantees that these quantities are finite, as is required by the condition of stationarity.

The z-transform

In the subsequent analysis, it will prove helpful to present the results in the notation of the z-transform. The z-transform of the infinite sequence $y(t) = \{y_t; t = 0, \pm 1, \pm 2, \ldots\}$ is defined by

$$(4.12) \qquad\qquad\qquad y(z) = \sum_{\tau=-\infty}^{\infty} y_t z^t.$$

Here z is a complex number which may be placed on the perimeter of the unit circle provided that the series converges. Thus $z = e^{-i\omega}$ with $\omega \in [0, 2\pi]$

If $y(t) = \mu_0 \varepsilon(t) + \mu_1 \varepsilon(t-1) + \cdots + \mu_q z^q \varepsilon(t-q) = \mu(L)\varepsilon(t)$ is a moving-average process, then the z-transform of the sequence of moving-average coefficients is the polynomial $\mu(z) = \mu_0 + \mu_1 z + \cdots + \mu_q z^q$ which has the same form as the operator $\mu(L)$.

The z-transform of a sequence of autocovariances is called the autocovariance generating function. For the moving-average process, this is given by

$$
\begin{aligned}
\gamma(z) &= \sigma_\varepsilon^2 \mu(z)\mu(z^{-1}) \\
&= \sigma_\varepsilon^2 \sum_i \mu_i z^i \sum_j \mu_j z^{-j} \\
&= \sigma_\varepsilon^2 \sum_i \sum_j \mu_i \mu_j z^{i-j} \\
&= \sum_\tau \left\{ \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau} \right\} z^\tau \quad ; \quad \tau = i - j \\
&= \sum_{\tau=-\infty}^{\infty} \gamma_\tau z^\tau .
\end{aligned}
$$

(4.13)

The final equality is by virtue of equation (10).

The Fourier Representation of a Sequence

According to the basic result of Fourier analysis, it is always possible to approximate an arbitrary analytic function defined over a finite interval of the real line, to any desired degree of accuracy, by a weighted sum of sine and cosine functions of harmonically increasing frequencies.

Similar results apply in the case of sequences, which may be regarded as functions mapping from the set of integers onto the real line. For a sample of T observations $y_0, \ldots, y_{T-1}$, it is possible to devise an expression in the form

(4.14)
$$
y_t = \sum_{j=0}^{n} \left\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \right\},
$$

wherein $\omega_j = 2\pi j/T$ is a multiple of the fundamental frequency $\omega_1 = 2\pi/T$. Thus, the elements of a finite sequence can be expressed exactly in terms of sines and cosines. This expression is called the Fourier decomposition of y_t and the set of coefficients $\{\alpha_j, \beta_j; j = 0, 1, \ldots, n\}$ are called the Fourier coefficients.

When T is even, we have $n = T/2$; and it follows that

(4.15)
$$
\begin{aligned}
\sin(\omega_0 t) &= \sin(0) = 0, \\
\cos(\omega_0 t) &= \cos(0) = 1, \\
\sin(\omega_n t) &= \sin(\pi t) = 0, \\
\cos(\omega_n t) &= \cos(\pi t) = (-1)^t .
\end{aligned}
$$

Therefore, equation (14) becomes

$$(4.16) \qquad y_t = \alpha_0 + \sum_{j=1}^{n-1} \left\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \right\} + \alpha_n (-1)^t.$$

When T is odd, we have $n = (T-1)/2$; and then equation (14) becomes

$$(4.17) \qquad y_t = \alpha_0 + \sum_{j=1}^{n} \left\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \right\}.$$

In both cases, there are T nonzero coefficients amongst the set $\{\alpha_j, \beta_j; j = 0, 1, \ldots, n\}$; and the mapping from the sample values to the co-efficients constitutes a one-to-one invertible transformation.

In equation (16), the frequencies of the trigonometric functions range from $\omega_1 = 2\pi/T$ to $\omega_n = \pi$; whereas, in equation (17), they range from $\omega_1 = 2\pi/T$ to $\omega_n = \pi(T-1)/T$. The frequency π is the so-called Nyquist frequency.

Although the process generating the data may contain components of frequencies higher than the Nyquist frequency, these will not be detected when it is sampled regularly at unit intervals of time. In fact, the effects on the process of components with frequencies in excess of the Nyquist value will be confounded with those whose frequencies fall below it.

To demonstrate this, consider the case where the process contains a component which is a pure cosine wave of unit amplitude and zero phase whose frequency ω lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$(4.18) \qquad \begin{aligned} \cos(\omega t) &= \cos\left\{ (2\pi - \omega^*)t \right\} \\ &= \cos(2\pi) \cos(\omega^* t) + \sin(2\pi) \sin(\omega^* t) \\ &= \cos(\omega^* t); \end{aligned}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* < \pi$ is described as the alias of $\omega > \pi$.

The Spectral Representation of a Stationary Process

By allowing the value of n in the expression (14) to tend to infinity, it is possible to express a sequence of indefinite length in terms of a sum of sine and cosine functions. However, in the limit as $n \to \infty$, the coefficients α_j, β_j tend to vanish; and therefore an alternative representation in terms of differentials is called for.

By writing $\alpha_j = dA(\omega_j)$, $\beta_j = dB(\omega_j)$ where $A(\omega)$, $B(\omega)$ are step functions with discontinuities at the points $\{\omega_j; j = 0, \ldots, n\}$, the expression (14) can be rendered as

$$(4.19) \qquad y_t = \sum_{j} \left\{ \cos(\omega_j t) dA(\omega_j) + \sin(\omega_j t) dB(\omega_j) \right\}.$$
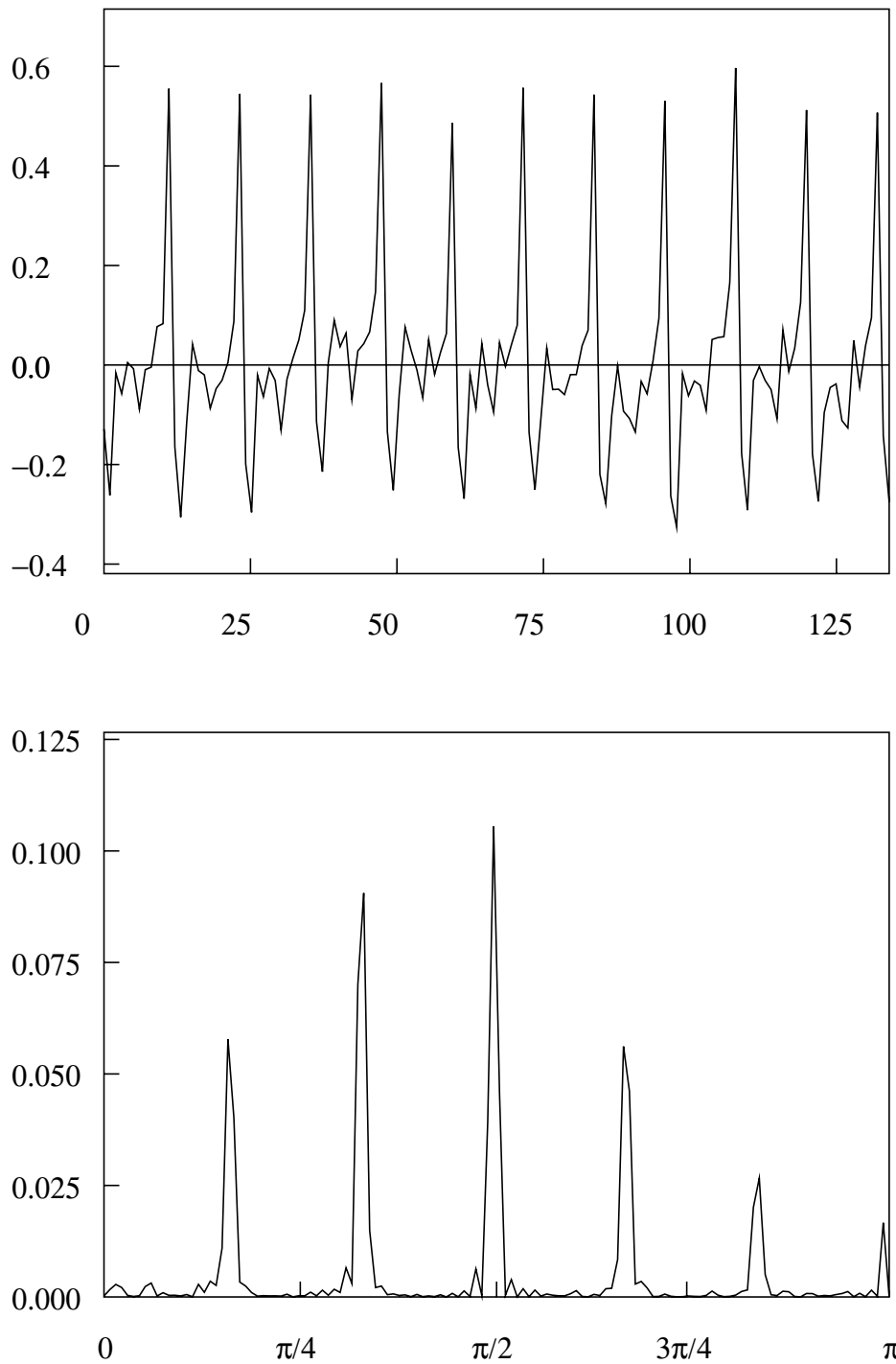
Figure 1. The graph of 134 observations on the monthly purchase of clothing after a logarithmic transformation and the removal of a linear trend together with the corresponding periodogram.

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In the limit, as $n \to \infty$, the summation is replaced by an integral to give the expression

$$(4.20) \qquad y(t) = \int_0^\pi \Big\{ \cos(\omega t) dA(\omega) + \sin(\omega t) dB(\omega) \Big\}.$$

Here, $\cos(\omega t)$ and $\sin(\omega t)$, and therefore $y(t)$, may be regarded as infinite sequences defined over the entire set of positive and negative integers.

Since $A(\omega)$ and $B(\omega)$ are discontinuous functions for which no derivatives exist, one must avoid using $\alpha(\omega)d\omega$ and $\beta(\omega)d\omega$ in place of $dA(\omega)$ and $dB(\omega)$. Moreover, the integral in equation (20) is a Fourier–Stieltjes integral.

In order to derive a statistical theory for the process that generates $y(t)$, one must make some assumptions concerning the functions $A(\omega)$ and $B(\omega)$. So far, the sequence $y(t)$ has been interpreted as a realisation of a stochastic process. If $y(t)$ is regarded as the stochastic process itself, then the functions $A(\omega)$, $B(\omega)$ must, likewise, be regarded as stochastic processes defined over the interval $[0, \pi]$. A single realisation of these processes now corresponds to a single realisation of the process $y(t)$.

The first assumption to be made is that the functions $A(\omega)$ and $B(\omega)$ represent a pair of stochastic processes of zero mean which are indexed on the continuous parameter ω. Thus

$$(4.21) \qquad E\big\{dA(\omega)\big\} = E\big\{dB(\omega)\big\} = 0.$$

The second and third assumptions are that the two processes are mutually uncorrelated and that non-overlapping increments within each process are uncorrelated. Thus

$$(4.22) \qquad \begin{aligned} E\big\{dA(\omega)dB(\lambda)\big\} &= 0 \quad \text{for all} \quad \omega, \lambda, \\ E\big\{dA(\omega)dA(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\big\{dB(\omega)dB(\lambda)\big\} &= 0 \quad \text{if} \quad \omega \neq \lambda. \end{aligned}$$

The final assumption is that the variance of the increments is given by

$$(4.23) \qquad \begin{aligned} V\big\{dA(\omega)\big\} = V\big\{dB(\omega)\big\} &= 2dF(\omega) \\ &= 2f(\omega)d\omega. \end{aligned}$$

We can see that, unlike $A(\omega)$ and $B(\omega)$, $F(\omega)$ is a continuous differentiable function. The function $F(\omega)$ and its derivative $f(\omega)$ are the spectral distribution function and the spectral density function, respectively.

In order to express equation (20) in terms of complex exponentials, we may define a pair of conjugate complex stochastic processes:

$$(4.24) \qquad \begin{aligned} dZ(\omega) &= \frac{1}{2}\big\{dA(\omega) - idB(\omega)\big\}, \\ dZ^*(\omega) &= \frac{1}{2}\big\{dA(\omega) + idB(\omega)\big\}. \end{aligned}$$

Also, we may extend the domain of the functions $A(\omega)$, $B(\omega)$ from $[0, \pi]$ to $[-\pi, \pi]$ by regarding $A(\omega)$ as an even function such that $A(-\omega) = A(\omega)$ and by regarding $B(\omega)$ as an odd function such that $B(-\omega) = -B(\omega)$. Then we have

$$(4.25) \qquad\qquad dZ^*(\omega) = dZ(-\omega).$$

From conditions under (22), it follows that

$$(4.26) \qquad \begin{aligned} E\{dZ(\omega)dZ^*(\lambda)\} &= 0 \quad \text{if} \quad \omega \neq \lambda, \\ E\{dZ(\omega)dZ^*(\omega)\} &= f(\omega)d\omega. \end{aligned}$$

These results may be used to reexpress equation (20) as

$$(4.27) \qquad \begin{aligned} y(t) &= \int_0^\pi \left\{ \frac{(e^{i\omega t} + e^{-i\omega t})}{2} dA(\omega) - i \frac{(e^{i\omega t} - e^{-i\omega t})}{2} dB(\omega) \right\} \\ &= \int_0^\pi \left\{ e^{i\omega t} \frac{\{dA(\omega) - idB(\omega)\}}{2} + e^{-i\omega t} \frac{\{dA(\omega) + idB(\omega)\}}{2} \right\} \\ &= \int_0^\pi \left\{ e^{i\omega t} dZ(\omega) + e^{-i\omega t} dZ^*(\omega) \right\}. \end{aligned}$$

When the integral is extended over the range $[-\pi, \pi]$, this becomes

$$(4.28) \qquad\qquad y(t) = \int_{-\pi}^{\pi} e^{i\omega t} dZ(\omega).$$

This is commonly described as the spectral representation of the process $y(t)$.

The Autocovariances and the Spectral Density Function

The sequence of the autocovariances of the process $y(t)$ may be expressed in terms of the spectrum of the process. From equation (28), it follows that the autocovariance y_t at lag $\tau = t - k$ is given by

$$(4.29) \qquad \begin{aligned} \gamma_\tau = C(y_t, y_k) &= E\left\{ \int_\omega e^{i\omega t} dZ(\omega) \int_\lambda e^{-i\lambda k} dZ(-\lambda) \right\} \\ &= \int_\omega \int_\lambda e^{i\omega t} e^{-i\lambda k} E\{dZ(\omega)dZ^*(\lambda)\} \\ &= \int_\omega e^{i\omega\tau} E\{dZ(\omega)dZ^*(\omega)\} \\ &= \int_\omega e^{i\omega\tau} f(\omega)d\omega. \end{aligned}$$
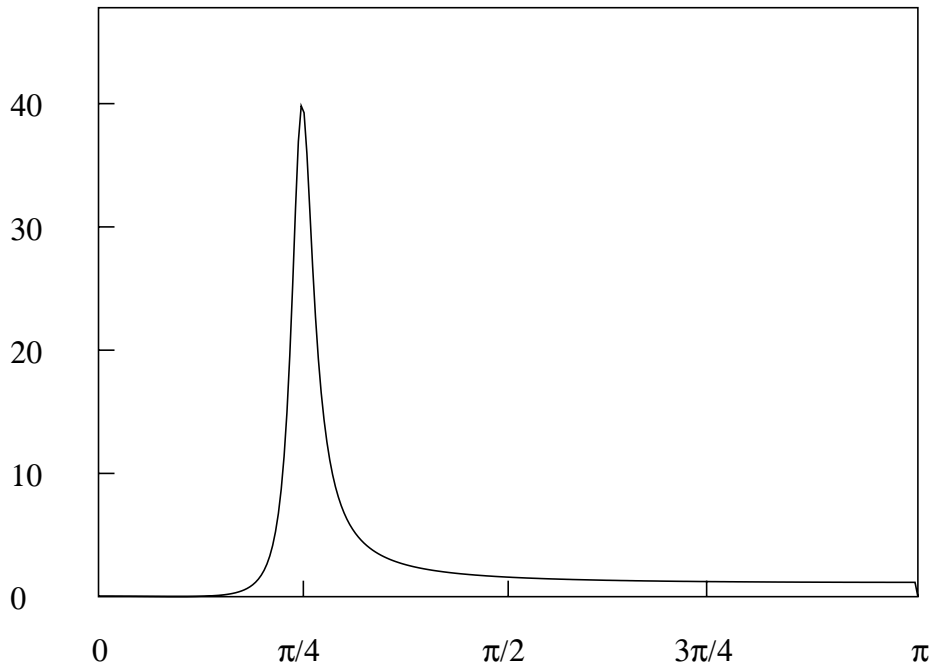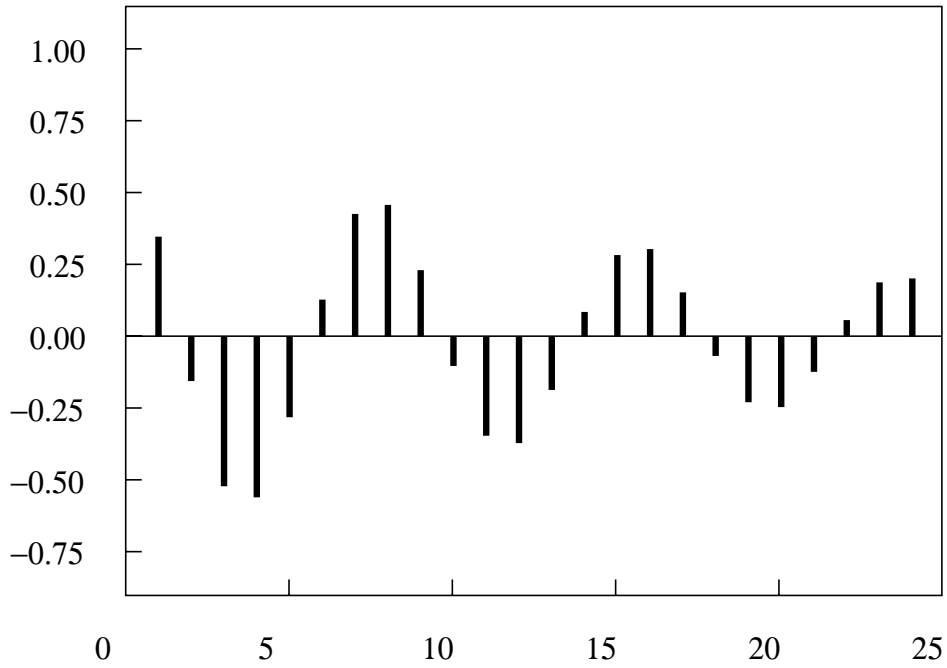
Figure 2. The theoretical autocorrelation function of the ARMA(2, 2) process $(1 - 1.344L + 0.902L^2)y(t) = (1 - 1.691L + 0.810L^2)\varepsilon(t)$ and (below) the corresponding spectral density function.

Here the final equalities are derived by using the results (25) and (26). This equation indicates that the Fourier transform of the spectrum is the autocovariance function.

The inverse mapping from the autocovariances to the spectrum is given by

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-i\omega\tau} \\
&= \frac{1}{2\pi} \left\{ \gamma_0 + 2 \sum_{\tau=1}^{\infty} \gamma_\tau \cos(\omega\tau) \right\}.
\end{aligned}
$$
(4.30)

This function is directly comparable to the periodogram of a data sequence which is defined under (2.41). However, the periodogram has T empirical autocovariances $c_0, \ldots, c_{T-1}$ in place of an indefinite number of theoretical autocovariances. Also, it differs from the spectrum by a scalar factor of 4π. In many texts, equation (30) serves as the primary definition of the spectrum.

To demonstrate the relationship which exists between equations (29) and (30), we may substitute the latter into the former to give

$$
\begin{aligned}
\gamma_\tau &= \int_{-\pi}^{\pi} e^{i\omega\tau} \left\{ \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \gamma_\tau e^{-i\omega\tau} \right\} d\omega \\
&= \frac{1}{2\pi} \sum_{\kappa=-\infty}^{\infty} \gamma_\kappa \int_{-\pi}^{\pi} e^{i\omega(\tau-\kappa)} d\omega.
\end{aligned}
$$
(4.31)

From the fact that

$$
\int_{-\pi}^{\pi} e^{i\omega(\tau-\kappa)} d\omega = \begin{cases} 2\pi, & \text{if } \kappa = \tau; \\ 0, & \text{if } \kappa \neq \tau, \end{cases}
$$
(4.32)

it can be seen that the RHS of the equation reduces to γ_τ. This serves to show that equations (29) and (30) do indeed represent a Fourier transform and its inverse.

The essential interpretation of the spectral density function is indicated by the equation

$$
\gamma_0 = \int_\omega f(\omega) d\omega,
$$
(4.33)

which comes from setting $\tau = 0$ in equation (29). This equation shows how the variance or 'power' of $y(t)$, which is γ_0, is attributed to the cyclical components of which the process is composed.

It is easy to see that a flat spectrum corresponds to the autocovariance function which characterises a white-noise process $\varepsilon(t)$. Let $f_\varepsilon = f_\varepsilon(\omega)$ be the flat spectrum. Then, from equation (30), it follows that

$$
\begin{aligned}
(4.34) \qquad \gamma_0 &= \int_{-\pi}^{\pi} f_\varepsilon(\omega) d\omega \\
&= 2\pi f_\varepsilon,
\end{aligned}
$$

and, from equation (29), it follows that

$$
\begin{aligned}
(4.35) \qquad \gamma_\tau &= \int_{-\pi}^{\pi} f_\varepsilon(\omega) e^{i\omega\tau} d\omega \\
&= f_\varepsilon \int_{-\pi}^{\pi} e^{i\omega\tau} d\omega \\
&= 0.
\end{aligned}
$$

These are the same as the conditions under (6) which have served to define a white-noise process. When the variance is denoted by σ_ε^2, the expression for the spectrum of the white-noise process becomes

$$
(4.36) \qquad f_\varepsilon(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}.
$$

Canonical Factorisation of the Spectral Density Function

Let $y(t)$ be a stationary stochastic process whose spectrum is $f_y(\omega)$. Since $f_y(\omega) \geq 0$, it is always possible to find a complex function $\mu(\omega)$ such that

$$
(4.37) \qquad f_y(\omega) = \frac{1}{2\pi} \mu(\omega)\mu^*(\omega).
$$

For a wide class of stochastic processes, the function $\mu(\omega)$ may be constructed in such a way that it can be expanded as a one-sided Fourier series:

$$
(4.38) \qquad \mu(\omega) = \sum_{j=0}^{\infty} \mu_j e^{-i\omega j}.
$$

On defining

$$
(4.39) \qquad dZ_\varepsilon(\omega) = \frac{dZ_y(\omega)}{\mu(\omega)},
$$

the spectral representation of the process $y(t)$ given in equation (28), may be rewritten as

$$(4.40) \qquad y(t) = \int_\omega e^{i\omega t} \mu(\omega) dZ_\varepsilon(\omega).$$

Expanding the expression of $\mu(\omega)$ and interchanging the order of integration and summation gives

$$
\begin{aligned}
y(t) &= \int_\omega e^{i\omega t} \left(\sum_j \mu_j e^{-i\omega j} \right) dZ_\varepsilon(\omega) \\
&= \sum_j \mu_j \left\{ \int_\omega e^{i\omega(t-j)} dZ_\varepsilon(\omega) \right\} \\
&= \sum_j \mu_j \varepsilon(t-j),
\end{aligned}
$$

(4.41)

where we have defined

$$(4.42) \qquad \varepsilon(t) = \int_\omega e^{i\omega t} dZ_\varepsilon(\omega).$$

The spectrum of $\varepsilon(t)$ is given by

$$
\begin{aligned}
E\{dZ_\varepsilon(\omega) dZ_\varepsilon^*(\omega)\} &= E \left\{ \frac{dZ_y(\omega) dZ_y^*(\omega)}{\mu(\omega)\mu^*(\omega)} \right\} \\
&= \frac{f_y(\omega)}{\mu(\omega)\mu^*(\omega)} \\
&= \frac{1}{2\pi}.
\end{aligned}
$$

(4.43)

Hence $\varepsilon(t)$ is identified as a white-noise process with unit variance. Therefore equation (38) represents a moving-average process; and what our analysis implies is that virtually every stationary stochastic process can be represented in this way.

The Frequency-Domain Analysis of Filtering

It is a straightforward matter to derive the spectrum of a process $y(t) = \mu(L)x(t)$ which is formed by mapping the process $x(t)$ through a linear filter.

Taking the spectral representation of the process $x(t)$ to be

$$(4.44) \qquad x(t) = \int_\omega e^{i\omega t} dZ_x(\omega),$$

we have

$$y(t) = \sum_j \mu_j x(t-j)$$

(4.45)
$$= \sum_j \mu_j \left\{ \int_\omega e^{i\omega(t-j)} dZ_x(\omega) \right\}$$

$$= \int_\omega e^{i\omega t} \left(\sum_j \mu_j e^{-i\omega j} \right) dZ_x(\omega).$$

On writing $\sum \mu_j e^{-i\omega j} = \mu(\omega)$, this becomes

$$y(t) = \int_\omega e^{i\omega t} \mu(\omega) dZ_x(\omega)$$

(4.46)
$$= \int_\omega e^{i\omega t} dZ_y(\omega).$$

It follows that the spectral density function $f_y(\omega)$ of the filtered process $y(t)$ is given by

(4.47)
$$\begin{aligned} f_y(\omega)d\omega &= E\{dZ_y(\omega)dZ_y^*(\omega)\} \\ &= \mu(\omega)\mu^*(\omega)E\{dZ_x(\omega)dZ_x^*(\omega)\} \\ &= |\mu(\omega)|^2 f_x(\omega)d\omega. \end{aligned}$$

In the case of the process defined in equation (7), where $y(t)$ is obtained by filtering a white-noise sequence, the result is specialised to give

(4.48)
$$\begin{aligned} f_y(\omega) &= |\mu(\omega)|^2 f_\varepsilon(\omega) \\ &= \frac{\sigma_\varepsilon^2}{2\pi} |\mu(\omega)|^2. \end{aligned}$$

Let $\mu(z) = \sum \mu_j z^j$ denote the z-transform of the sequence $\{\mu_j\}$. Then

(4.49)
$$\begin{aligned} |\mu(z)|^2 &= \mu(z)\mu(z^{-1}) \\ &= \sum_\tau \sum_j \mu_j \mu_{j+\tau} z^\tau. \end{aligned}$$

It follows that, when $z = e^{-i\omega}$, equation (48) can be written as

(4.50)
$$\begin{aligned} f_y(\omega) &= \frac{\sigma_\varepsilon^2}{2\pi} \mu(z)\mu(z^{-1}) \\ &= \frac{1}{2\pi} \sum_\tau \left\{ \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau} \right\} z^\tau. \end{aligned}$$

But, according to equation (10), $\gamma_\tau = \sigma_\varepsilon^2 \sum_j \mu_j \mu_{j+\tau}$ is the autocovariance of lag τ of the process $y(t)$. Therefore, the function $f_y(\omega)$ can be written as

(4.51)
$$f_y(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} e^{-i\omega\tau} \gamma_\tau$$
$$= \frac{1}{2\pi} \left\{ \gamma_0 + 2 \sum_{\tau=1}^{\infty} \gamma_\tau \cos(\omega\tau) \right\},$$

which indicates that the spectral density function is the Fourier transform of the autocovariance function of the filtered sequence. This is known as the Wiener–Khintchine theorem. The importance of this theorem is that it provides a link between the time domain and the frequency domain.

The Gain and Phase

The complex-valued function $\mu(\omega)$, which is entailed in the process of linear filtering, can be written as

(4.52)
$$\mu(\omega) = |\mu(\omega)| e^{-i\theta(\omega)}.$$

where

(4.53)
$$|\mu(\omega)|^2 = \left\{ \sum_{j=0}^{\infty} \mu_j \cos(\omega j) \right\}^2 + \left\{ \sum_{j=0}^{\infty} \mu_j \sin(\omega j) \right\}^2$$
$$\theta(\omega) = \arctan \left\{ \frac{\sum \mu_j \sin(\omega j)}{\sum \mu_j \cos(\omega j)} \right\}.$$

The function $|\mu(\omega)|$, which is described as the gain of the filter, indicates the extent to which the amplitude of the cyclical components of which $x(t)$ is composed are altered in the process of filtering.

The function $\theta(\omega)$, which is described as the phase displacement and which gives a measure in radians, indicates the extent to which the cyclical components are displaced along the time axis.

The substitution of expression (52) in equation (46) gives

(4.54)
$$y(t) = \int_{-\pi}^{\pi} e^{i\{\omega t - \theta(\omega)\}} |\mu(\omega)| dZ_x(\omega).$$

The importance of this equation is that it summarises the two effects of the filter.