LECTURE 2

Seasons and Cycles in Time Series

Cycles of a regular nature are often encountered in physics and engineering. Consider a point moving with constant speed in a circle of radius ρ. The point might be the axis of the 'big end' of a connecting rod which joins a piston to a flywheel. Let time t be reckoned from an instant when the radius joining the point to the centre is at an angle of θ below the horizontal. If the point is projected onto the horizontal axis, then the distance of the projection from the centre is given by

$$x = \rho \cos(\omega t - \theta). \tag{2.1}$$

The movement of the projection back and forth along the horizontal axis is described as simple harmonic motion.

The parameters of the function are as follows:

ρ is the amplitude,

ω is the angular velocity or frequency and

θ is the phase displacement.

The angular velocity is a measure in radians per unit period. The quantity $2\pi/\omega$ measures the period of the cycle. The phase displacement, also measured in radians, indicates the extent to which the cosine function has been displaced by a shift along the time axis. Thus, instead of the peak of the function occurring at time $t = 0$, as it would with an ordinary cosine function, it now occurs a time $t = \theta/\omega$.

Using the compound-angle formula $\cos(A - B) = \cos A \cos B + \sin A \sin B$, we can rewrite equation (1) as

$$\begin{aligned} x &= \rho \cos\theta \cos(\omega t) + \rho \sin\theta \sin(\omega t) \\ &= \alpha \cos(\omega t) + \beta \sin(\omega t), \end{aligned} \tag{2.2}$$

with

$$\alpha = \rho\cos\theta, \qquad \beta = \rho\sin\theta \qquad \text{and} \qquad \alpha^2 + \beta^2 = \rho^2. \tag{2.3}$$

Extracting a Regular Cyclical Component

A cyclical component which is concealed beneath other motions may be extracted from a data sequence by a straightforward application of the method of linear regression. An equation may be written in the form of

$$y_t = \alpha c_t(\omega) + \beta s_t(\omega) + e_t; \quad t = 0, \ldots, T-1, \tag{2.4}$$

where $c_t(\omega) = \cos(\omega t)$ and $s_t(\omega) = \sin(\omega t)$. To avoid the need for an intercept term, the values of the dependent variable should be deviations about a mean value. In matrix terms, equation (4) becomes

$$y = \begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + e, \tag{2.5}$$

where $c = [c_0, \ldots, c_{T-1}]'$ and $s = [s_0, \ldots, s_{T-1}]'$ and $e = [e_0, \ldots, e_{T-1}]'$ are vectors of T elements. The parameters α, β can be found by running regressions for a wide range of values of ω and by selecting the regression which delivers the lowest value for the residual sum of squares.

Such a technique may be used for extracting a seasonal component from an economic time series; and, in that case, we know in advance what value to give to ω. For the seasonality of economic activities is related, ultimately, to the near-perfect regularities of the solar system which are reflected in the annual calender.

It may be unreasonable to expect that an idealised seasonal cycle can be represented by a simple sinusoidal function. However, wave forms of a more complicated nature may be synthesised by employing a series of sine and cosine functions whose frequencies are integer multiples of the fundamental seasonal frequency. If there are $s = 2n$ observations per annum, then a general model for a seasonal fluctuation would comprise the frequencies

$$\omega_j = \frac{2\pi j}{s}, \quad j = 0, \ldots, n = \frac{s}{2}, \tag{2.6}$$

which are equally spaced in the interval $[0, \pi]$. Such a series of frequencies is described as an harmonic scale.

A model of seasonal fluctuation comprising the full set of harmonically-related frequencies would take the form of

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\} + e_t, \tag{2.7}$$

where e_t is a residual element which might represent an irregular white-noise component in the process underlying the data.

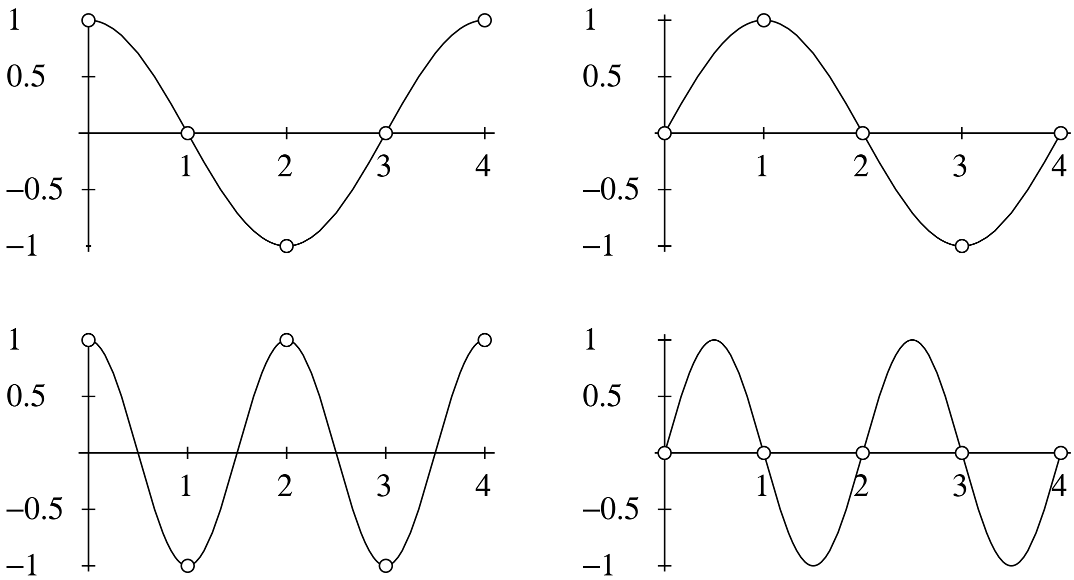

Figure 1. Trigonometrical functions, of frequencies $\omega_1 = \pi/2$ and $\omega_2 = \pi$, associated with a quarterly model of a seasonal fluctuation.

At first sight, it appears that there are $s + 2$ components in the sum. However, when s is even, we have

$$\begin{aligned} \sin(\omega_0 t) &= \sin(0) = 0, \\ \cos(\omega_0 t) &= \cos(0) = 1, \\ \sin(\omega_n t) &= \sin(\pi t) = 0, \\ \cos(\omega_n t) &= \cos(\pi t) = (-1)^t. \end{aligned} \tag{2.8}$$

Therefore there are only s nonzero coefficients to be determined.

This simple seasonal model is illustrated adequately by the case of quarterly data. Matters are no more complicated in the case of monthly data. When there are four observations per annum, we have $\omega_0 = 0$, $\omega_1 = \pi/2$ and $\omega_2 = \pi$; and equation (7) assumes the form of

$$y_t = \alpha_0 + \alpha_1 \cos\left(\frac{\pi t}{2}\right) + \beta_1 \sin\left(\frac{\pi t}{2}\right) + \alpha_2(-1)^t + e_t. \tag{2.9}$$

If the four seasons are indexed by $j = 0, \ldots, 3$, then the values from the year τ can be represented by the following matrix equation:

$$\begin{bmatrix} y_{\tau 0} \\ y_{\tau 1} \\ y_{\tau 2} \\ y_{\tau 3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_1 \\ \alpha_2 \end{bmatrix} + \begin{bmatrix} e_{\tau 0} \\ e_{\tau 1} \\ e_{\tau 2} \\ e_{\tau 3} \end{bmatrix}. \tag{2.10}$$

It will be observed that the vectors of the matrix are mutually orthogonal.

When the data consist of $T = 4p$ observations which span p years, the coefficients of the equation are given by

$$\begin{aligned}
\alpha_0 &= \frac{1}{T}\sum_{t=0}^{T-1} y_t,\\
\alpha_1 &= \frac{2}{T}\sum_{\tau=1}^{p}(y_{\tau 0} - y_{\tau 2}),\\
\beta_1 &= \frac{2}{T}\sum_{\tau=1}^{p}(y_{\tau 1} - y_{\tau 3}),\\
\alpha_2 &= \frac{1}{T}\sum_{\tau=1}^{p}(y_{\tau 0} - y_{\tau 1} + y_{\tau 2} - y_{\tau 3}).
\end{aligned} \tag{2.11}$$

It is the mutual orthogonality of the vectors of 'explanatory' variables which accounts for the simplicity of these formulae.

An alternative model of seasonality, which is used more often by econometricians, assigns an individual dummy variable to each season. Thus, in place of equation (10), we may take

$$\begin{bmatrix} y_{\tau 0}\\ y_{\tau 1}\\ y_{\tau 2}\\ y_{\tau 3}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\delta_0\\ \delta_1\\ \delta_2\\ \delta_3\end{bmatrix} + \begin{bmatrix} e_{\tau 0}\\ e_{\tau 1}\\ e_{\tau 2}\\ e_{\tau 3}\end{bmatrix}, \tag{2.12}$$

where

$$\delta_j = \frac{4}{T}\sum_{\tau=1}^{p} y_{\tau j}, \quad \text{for} \quad j = 0, \ldots, 3. \tag{2.13}$$

A comparison of equations (10) and (12) establishes the mapping from the coefficients of the trigonometrical functions to the coefficients of the dummy variables. The inverse mapping is

$$\begin{bmatrix}\alpha_0\\ \alpha_1\\ \beta_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & \frac{1}{2} & 0 & -\frac{1}{2}\\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4}\end{bmatrix}\begin{bmatrix}\delta_0\\ \delta_1\\ \delta_2\\ \delta_3\end{bmatrix}. \tag{2.14}$$

Another way of parametrising the model of seasonality is to adopt the following form:

$$\begin{bmatrix} y_{\tau 0}\\ y_{\tau 1}\\ y_{\tau 2}\\ y_{\tau 3}\end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\phi\\ \gamma_0\\ \gamma_1\\ \gamma_2\end{bmatrix} + \begin{bmatrix} e_{\tau 0}\\ e_{\tau 1}\\ e_{\tau 2}\\ e_{\tau 3}\end{bmatrix}. \tag{2.15}$$

This scheme is unbalanced in that it does not treat each season in the same manner. An attempt might be made to correct this feature by adding to the matrix an extra column with a unit at the bottom and with zeros elsewhere and by introducing an accompanying parameter γ_3. However, the columns of the resulting matrix will be linearly dependent; and this will make the parameters indeterminate unless an additional constraint is imposed which sets $\gamma_0 + \cdots + \gamma_3 = 0$.

The problem highlights a difficulty which might arise if either of the schemes under (10) or (12) were fitted to the data by multiple regression in the company of a polynomial $\phi(t) = \phi_0 + \phi_1 t + \cdots + \phi_p t^p$ designed to capture a trend. To make such a regression viable, one would have to eliminate the intercept parameter ϕ_0.

Irregular Cycles

Whereas it seems reasonable to model a seasonal fluctuation in terms of trigonometrical functions, it is difficult to accept that other cycles in economic activity should have such regularity.

A classic expression of skepticism was made by Slutsky [19] in a famous article of 1927:

> Suppose we are inclined to believe in the reality of the strict periodicity of the business cycle, such, for example, as the eight-year period postulated by Moore. Then we should encounter another difficulty. Wherein lies the source of this regularity? What is the mechanism of causality which, decade after decade, reproduces the same sinusoidal wave which rises and falls on the surface of the social ocean with the regularity of day and night?

It seems that something other than a perfectly regular sinusoidal component is required to model the secular fluctuations of economic activity which are described as business cycles.

To obtain a model for a seasonal fluctuation, it has been enough to modify the equation of harmonic motion by superimposing a disturbance term which affects the amplitude. To generate a cycle which is more fundamentally affected by randomness, we must construct a model which has random effects in both the phase and the amplitude.

To begin, let us imagine, once more, a point on the circumference of a circle of radius ρ which is travelling with an angular velocity of ω. At the instant $t = 0$, when the point makes a positive angle of θ with the horizontal axis, the coordinates are given by

$$(\alpha, \beta) = (\rho \cos\theta, \rho \sin\theta). \tag{2.16}$$

To find the coordinates of the point after it has rotated through an angle of ω in one period of time, we may rotate the component vectors $(\alpha, 0)$ and $(0, \beta)$ separately and add them. The rotation of the components is depicted as follows:

$$(2.17)\qquad \begin{aligned} (\alpha, 0) &\xrightarrow{\omega} (\alpha\cos\omega, \alpha\sin\omega), \\ (0, \beta) &\xrightarrow{\omega} (-\beta\sin\omega, \beta\cos\omega). \end{aligned}$$

Their addition gives

$$(2.18)\qquad (\alpha, \beta) \xrightarrow{\omega} (y, z) = (\alpha\cos\omega - \beta\sin\omega, \alpha\sin\omega + \beta\cos\omega).$$

In matrix terms, the transformation becomes

$$(2.19)\qquad \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

To find the values of the coordinates at a time which is an integral number of periods ahead, we may transform the vector $[y', z']'$ by premultiplying it the appropriate number of times by the matrix of the rotation. Alternatively, we may replace ω in equation (19) by whatever angle will be reached at the time in question. In effect, equation (19) specifies the horizontal and vertical components of a circular motion which amount to a pair of synchronous harmonic motions.

To introduce the appropriate irregularities into the motion, we may add a random disturbance term to each of its components. The discrete-time equation of the resulting motion may be expressed as follows:

$$(2.20)\qquad \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \upsilon_t \\ \zeta_t \end{bmatrix}.$$

Now the character of the motion is radically altered. There is no longer any bound on the amplitudes which the components might acquire in the long run; and there is, likewise, a tendency for the phases of their cycles to drift without limit. Nevertheless, in the absence of uncommonly large disturbances, the trajectories of y and z are liable, in a limited period, to resemble those of the simple harmonic motions.

It is easy to decouple the equations of y and z. The first of the equations within the matrix expression can be written as

$$(2.21)\qquad y_t = cy_{t-1} - sz_{t-1} + \upsilon_t.$$

The second equation may be lagged by one period and rearranged to give

$$(2.22)\qquad z_{t-1} - cz_{t-2} = sy_{t-2} + \zeta_{t-1}.$$

By taking $cy_{t-1} = c^2y_{t-2} - csz_{t-2} + \upsilon_{t-2}$ from equation (21) and by using equation (22) to eliminate the values of z, we get

$$
\begin{aligned}
y_t - cy_{t-1} &= cy_{t-1} - c^2y_{t-2} - sz_{t-1} + csz_{t-2} + \upsilon_t - c\upsilon_{t-1} \\
&= cy_{t-1} - c^2y_{t-2} - s^2y_{t-2} - s\zeta_{t-1} + \upsilon_t - c\upsilon_{t-1}.
\end{aligned} \tag{2.23}
$$

If we use the result that $y_{t-2}\cos^2 + y_{t-2}\sin^2 = y_{t-2}$ and if we collect the disturbances to form a new variable $\varepsilon_t = \upsilon_t - s\zeta_{t-1} - c\upsilon_{t-1}$, then we can rearrange the second equality to give

$$
y_t = 2\cos\omega y_{t-1} - y_{t-2} + \varepsilon_t. \tag{2.24}
$$

Here it is not true in general that the sequence of disturbances $\{\varepsilon_t\}$ will be white noise. However, if we specify that, within equation (20),

$$
\begin{bmatrix} \upsilon_t \\ \zeta_t \end{bmatrix} = \begin{bmatrix} -\sin\omega \\ \cos\omega \end{bmatrix} \eta_t, \tag{2.25}
$$

where $\{\eta_t\}$ is a white-noise sequence, then the lagged terms within ε_t will cancel leaving a sequence whose elements are mutually uncorrelated.

A sequence generated by equation (24) when $\{\varepsilon_t\}$ is a white-noise sequence is depicted in Figure 2.

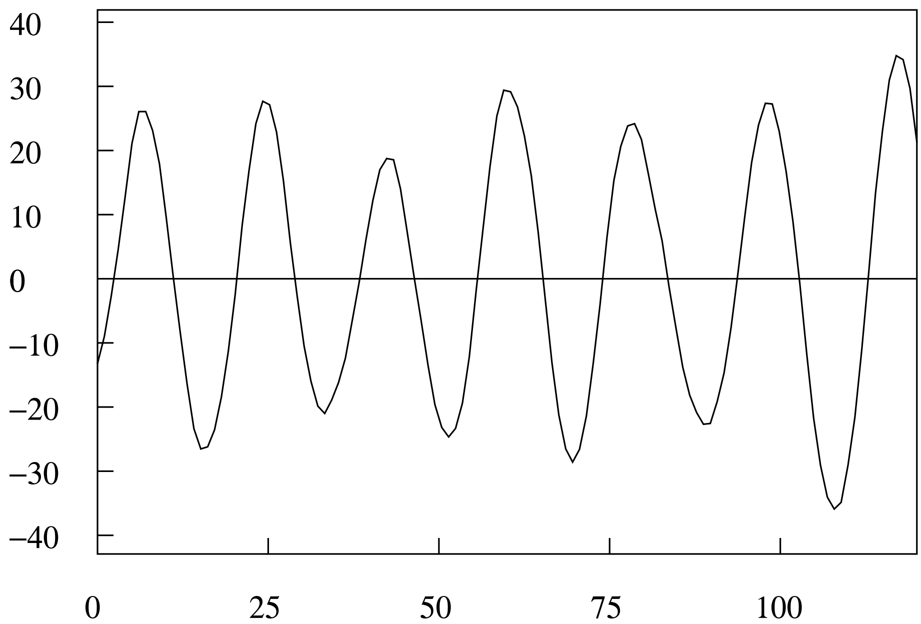

Figure 2. A quasi-cyclical sequence generated by the equation $y_t = 2\cos\omega y_{t-1} - y_{t-2} + \varepsilon_t$ when $\omega = 20°$.

It is interesting to recognise that equation (24) becomes the equation of a second-order random walk in the case where $\omega = 0$. The second-order random walk gives rise to trends which can remain virtually linear over considerable periods.

Whereas there is little difficulty in understanding that an accumulation of purely random disturbances can give rise to a linear trend, there is often surprise at the fact that such disturbances can also generate cycles which are more or less regular. An understanding of this phenomenon can be reached by considering a physical analogy. One such analogy, which is very apposite, was provided by Yule whose article of 1927 introduced the concept of a second-order autoregressive process of which equation (24) is a limiting case. Yules's purpose was to explain, in terms of random causes, a cycle of roughly 11 years which characterises the Wolfer sunspot index.

Yule invited his readers to imagine a pendulum attached to a recording device. Any deviations from perfectly harmonic motion which might be recorded must be the result of superimposed errors of observation which could be all but eliminated if a long sequence of observations were subjected to a regression analysis.

> The recording apparatus is left to itself and unfortunately boys get into the room and start pelting the pendulum with peas, sometimes from one side and sometimes from the other. The motion is now affected not by *superposed fluctuations* but by true *disturbances*, and the effect on the graph will be of an entirely different kind. The graph will remain surprisingly smooth, but amplitude and phase will vary continuously.

The phenomenon described by Yule is due to the inertia of the pendulum. In the short term, the impacts of the peas impart very little energy to the system compared with the sum of its kinetic and potential energies at any point in time. However, on taking a longer view, we can see that, in the absence of clock weights, the system is driven by the impacts alone.

The Fourier Decomposition of a Time Series

In spite of the notion that a regular trigonometrical function is an inappropriate means for modelling an economic cycle other than a seasonal fluctuation, there are good reasons to persist with the business of explaining a data sequence in terms of such functions.

The Fourier decomposition of a series is a matter of explaining the series entirely as a composition of sinusoidal functions. Thus it is possible to represent the generic element of the sample as

$$y_t = \sum_{j=0}^{n} \Big\{ \alpha_j \cos(\omega_j t) + \beta_j \sin(\omega_j t) \Big\}. \tag{2.26}$$

Assuming that $T = 2n$ is even, this sum comprises T functions whose frequencies

$$\omega_j = \frac{2\pi j}{T}, \quad j = 0, \ldots, n = \frac{T}{2} \tag{2.27}$$

are at equally spaced points in the interval $[0, \pi]$.

As we might infer from our analysis of a seasonal fluctuation, there are as many nonzeros elements in the sum under (26) as there are data points, for the reason that two of the functions within the sum—namely $\sin(\omega_0 t) = \sin(0)$ and $\sin(\omega_n t) = \sin(\pi t)$—are identically zero. It follows that the mapping from the sample values to the coefficients constitutes a one-to-one invertible transformation. The same conclusion arises in the slightly more complicated case where T is odd.

The angular velocity $\omega_j = 2\pi j/T$ relates to a pair of trigonometrical components which accomplish j cycles in the T periods spanned by the data. The highest velocity $\omega_n = \pi$ corresponds to the so-called Nyquist frequency. If a component with a frequency in excess of π were included in the sum in (26), then its effect would be indistinguishable from that of a component with a frequency in the range $[0, \pi]$

To demonstrate this, consider the case of a pure cosine wave of unit amplitude and zero phase whose frequency ω lies in the interval $\pi < \omega < 2\pi$. Let $\omega^* = 2\pi - \omega$. Then

$$\begin{aligned} \cos(\omega t) &= \cos\left\{(2\pi - \omega^*)t\right\} \\ &= \cos(2\pi)\cos(\omega^* t) + \sin(2\pi)\sin(\omega^* t) \\ &= \cos(\omega^* t); \end{aligned} \tag{2.28}$$

which indicates that ω and ω^* are observationally indistinguishable. Here, $\omega^* \in [0, \pi]$ is described as the alias of $\omega > \pi$.

For an illustration of the problem of aliasing, let us imagine that a person observes the sea level at 6am. and 6pm. each day. He should notice a very gradual recession and advance of the water level; the frequency of the cycle being $f = 1/28$ which amounts to one tide in 14 days. In fact, the true frequency is $f = 1 - 1/28$ which gives 27 tides in 14 days. Observing the sea level every six hours should enable him to infer the correct frequency.

Calculation of the Fourier Coefficients

For heuristic purposes, we can imagine calculating the Fourier coefficients using an ordinary regression procedure to fit equation (26) to the data. In this case, there would be no regression residuals, for the reason that we are 'estimating' a total of T coefficients from T data points; so we are actually solving a set of T linear equations in T unknowns.

A reason for not using a multiple regression procedure is that, in this case, the vectors of 'explanatory' variables are mutually orthogonal. Therefore T applications of a univariate regression procedure would be appropriate to our purpose.

Let $c_j = [c_{0j}, \ldots, c_{T-1,j}]'$ and $s_j = [s_{0,j}, \ldots, s_{T-1,j}]'$ represent vectors of T values of the generic functions $\cos(\omega_j t)$ and $\sin(\omega_j t)$ respectively. Then there are the following orthogonality conditions:

$$\begin{aligned} c_i'c_j &= 0 \quad \text{if} \quad i \neq j, \\ s_i's_j &= 0 \quad \text{if} \quad i \neq j, \\ c_i's_j &= 0 \quad \text{for all} \quad i, j. \end{aligned} \tag{2.29}$$

In addition, there are the following sums of squares:

$$\begin{aligned} c_0'c_0 &= c_n'c_n = T, \\ s_0's_0 &= s_n's_n = 0, \\ c_j'c_j &= s_j's_j = \frac{T}{2}. \end{aligned} \tag{2.30}$$

The 'regression' formulae for the Fourier coefficients are therefore

$$\alpha_0 = (i'i)^{-1}i'y = \frac{1}{T}\sum_t y_t = \bar{y}, \tag{2.31}$$

$$\alpha_j = (c_j'c_j)^{-1}c_j'y = \frac{2}{T}\sum_t y_t \cos\omega_i t, \tag{2.32}$$

$$\beta_j = (s_j's_j)^{-1}s_j'y = \frac{2}{T}\sum_t y_t \sin\omega_j t. \tag{2.33}$$

By pursuing the analogy of multiple regression, we can understand that there is a complete decomposition of the sum of squares of the elements of y which is given by

$$y'y = \alpha_0^2 i'i + \sum_j \alpha_j^2 c_j'c_j + \sum_j \beta_j^2 s_j's_j. \tag{2.34}$$

Now consider writing $\alpha_0^2 i'i = \bar{y}^2 i'i = \bar{y}'\bar{y}$ where $\bar{y}' = [\bar{y}, \ldots, \bar{y}]$ is the vector whose repeated element is the sample mean $\bar{y}$. It follows that $y'y - \alpha_0^2 i'i = y'y - \bar{y}'\bar{y} = (y-\bar{y})'(y-\bar{y})$. Therefore we can rewrite the equation as

$$(y-\bar{y})'(y-\bar{y}) = \frac{T}{2}\sum_j \left\{\alpha_j^2 + \beta_j^2\right\} = \frac{T}{2}\sum_j \rho_j^2, \tag{2.35}$$

and it follows that we can express the variance of the sample as

$$(2.36)\qquad \begin{aligned} \frac{1}{T}\sum_{t=0}^{T-1}(y_t-\bar{y})^2 &= \frac{1}{2}\sum_{j=1}^{n}(\alpha_j^2+\beta_j^2) \\ &= \frac{2}{T^2}\sum_j\left\{\left(\sum_t y_t\cos\omega_j t\right)^2+\left(\sum_t y_t\sin\omega_j t\right)^2\right\}. \end{aligned}$$

The proportion of the variance which is attributable to the component at frequency ω_j is $(\alpha_j^2+\beta_j^2)/2=\rho_j^2/2$, where ρ_j is the amplitude of the component.

The number of the Fourier frequencies increases at the same rate as the sample size T. Therefore, if the variance of the sample remains finite, and if there are no regular harmonic components in the process generating the data, then we can expect the proportion of the variance attributed to the individual frequencies to decline as the sample size increases. If there is such a regular component within the process, then we can expect the proportion of the variance attributable to it to converge to a finite value as the sample size increases.

In order provide a graphical representation of the decomposition of the sample variance, we must scale the elements of equation (36) by a factor of T. The graph of the function $I(\omega_j)=(T/2)(\alpha_j^2+\beta_j^2)$ is know as the periodogram.

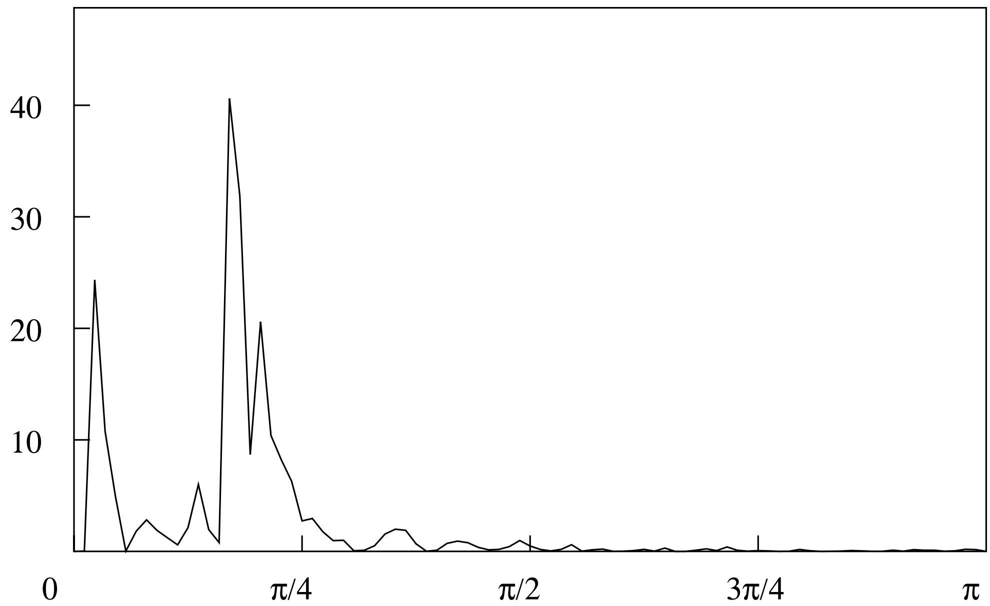

Figure 3. The periodogram of Wolfer's Sunspot Numbers 1749–1924.

There are many impressive examples where the estimation of the periodogram has revealed the presence of regular harmonic components in a data series which might otherwise have passed undetected. One of the best-know examples concerns the analysis of the brightness or magnitude of the star T. Ursa Major. It was shown by Whittaker and Robinson in 1924 that this series could be described almost completely in terms of two trigonometrical functions with periods of 24 and 29 days.

The attempts to discover underlying components in economic time-series have been less successful. One application of periodogram analysis which was a notorious failure was its use by William Beveridge in 1921 and 1923 to analyse a long series of European wheat prices. The periodogram had so many peaks that at least twenty possible hidden periodicities could be picked out, and this seemed to be many more than could be accounted for by plausible explanations within the realms of economic history.

Such findings seem to diminish the importance of periodogram analysis in econometrics. However, the fundamental importance of the periodogram is established once it is recognised that it represents nothing less than the Fourier transform of the sequence of empirical autocovariances.

The Empirical Autocovariances

A natural way of representing the serial dependence of the elements of a data sequence is to estimate their autocovariances. The empirical autocovariance of lag τ is defined by the formula

$$c_\tau = \frac{1}{T}\sum_{t=\tau}^{T-1}(y_t - \bar{y})(y_{t-\tau} - \bar{y}). \tag{2.37}$$

The empirical autocorrelation of lag τ is defined by $r_\tau = c_\tau / c_0$ where c_0, which is formally the autocovariance of lag 0, is the variance of the sequence. The autocorrelation provides a measure of the relatedness of data points separated by τ periods which is independent of the units of measurement.

It is straightforward to establish the relationship between the periodogram and the sequence of autocovariances.

The periodogram may be written as

$$I(\omega_j) = \frac{2}{T}\left[\left\{\sum_{t=0}^{T-1}\cos(\omega_j t)(y_t - \bar{y})\right\}^2 + \left\{\sum_{t=0}^{T-1}\sin(\omega_j t)(y_t - \bar{y})\right\}^2\right]. \tag{2.38}$$

The identity $\sum_t \cos(\omega_j t)(y_t - \bar{y}) = \sum_t \cos(\omega_j t) y_t$ follows from the fact that, by construction, $\sum_t \cos(\omega_j t) = 0$ for all j. Expanding the expression in (38) gives

(2.39)
$$\begin{aligned} I(\omega_j) =& \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j t)\cos(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\} \\ &+ \frac{2}{T}\left\{\sum_t\sum_s \sin(\omega_j t)\sin(\omega_j s)(y_t - \bar{y})(y_s - \bar{y})\right\}, \end{aligned}$$

and, by using the identity $\cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A - B)$, we can rewrite this as

(2.40)
$$I(\omega_j) = \frac{2}{T}\left\{\sum_t\sum_s \cos(\omega_j[t-s])(y_t - \bar{y})(y_s - \bar{y})\right\}.$$

Next, on defining $\tau = t - s$ and writing $c_\tau = \sum_t (y_t - \bar{y})(y_{t-\tau} - \bar{y})/T$, we can reduce the latter expression to

(2.41)
$$I(\omega_j) = 2\sum_{\tau=1-T}^{T-1} \cos(\omega_j \tau) c_\tau,$$

which is a Fourier transform of the sequence of empirical autocovariances.

An Appendix on Harmonic Cycles

Lemma 1. Let $\omega_j = 2\pi j/T$ where $j \in \{0, 1, \ldots, T/2\}$ if T is even and $j \in \{0, 1, \ldots, (T-1)/2\}$ if T is odd. Then

$$\sum_{t=0}^{T-1} \cos(\omega_j t) = \sum_{t=0}^{T-1} \sin(\omega_j t) = 0.$$

Proof. By Euler's equations, we have

$$\sum_{t=0}^{T-1} \cos(\omega_j t) = \frac{1}{2}\sum_{t=0}^{T-1} \exp(i2\pi jt/T) + \frac{1}{2}\sum_{t=0}^{T-1} \exp(-i2\pi jt/T).$$

By using the formula $1 + \lambda + \cdots + \lambda^{T-1} = (1 - \lambda^T)/(1 - \lambda)$, we find that

$$\sum_{t=0}^{T-1} \exp(i2\pi jt/T) = \frac{1 - \exp(i2\pi j)}{1 - \exp(i2\pi j/T)}.$$

But $\exp(i2\pi j) = \cos(2\pi j) + i\sin(2\pi j) = 1$, so the numerator in the expression above is zero, and hence $\sum_t \exp(i2\pi j/T) = 0$. By similar means, we can show that $\sum_t \exp(-i2\pi j/T) = 0$; and, therefore, it follows that $\sum_t \cos(\omega_j t) = 0$. An analogous proof shows that $\sum_t \sin(\omega_j t) = 0$.

Lemma 2. Let $\omega_j = 2\pi j/T$ where $j \in 0, 1, \ldots, T/2$ if T is even and $j \in 0, 1, \ldots, (T-1)/2$ if T is odd. Then

$$(a) \qquad \sum_{t=0}^{T-1} \cos(\omega_j t)\cos(\omega_k t) = \begin{cases} 0, & \text{if } j \neq k; \\ \frac{T}{2}, & \text{if } j = k. \end{cases}$$

$$(b) \qquad \sum_{t=0}^{T-1} \sin(\omega_j t)\sin(\omega_k t) = \begin{cases} 0, & \text{if } j \neq k; \\ \frac{T}{2}, & \text{if } j = k. \end{cases}$$

$$(c) \qquad \sum_{t=0}^{T-1} \cos(\omega_j t)\sin(\psi_k t) = 0 \qquad \text{if} j \neq k.$$

Proof. From the formula $\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$ we have

$$\begin{aligned}\sum_{t=0}^{T-1} \cos(\omega_j t)\cos(\omega_k t) &= \frac{1}{2}\sum \{\cos([\omega_j + \omega_k]t) + \cos([\omega_j - \psi_k]t)\} \\ &= \frac{1}{2}\sum_{t=0}^{T-1}\{\cos(2\pi[j+k]t/T) + \cos(2\pi[j-k]t/T)\}.\end{aligned}$$

We find, in consequence of Lemma 1, that if $j \neq k$, then both terms on the RHS vanish, and thus we have the first part of (a). If $j = k$, then $\cos(2\pi[j-k]t/T) = \cos 0 = 1$ and so, whilst the first term vanishes, the second terms yields the value of T under summation. This gives the second part of (a).

The proofs of (b) and (c) follow along similar lines.

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