The Two-Channel Quadrature Mirror Filter Bank

Consider the following way of processing a signal. First, the signal is transmitted through two separate branches containing a low pass filter H_0 and a high pass filter H_1. Then the filtered signals are down sampled by selecting alternate data points, indexed by even integers, and by discarding the points indexed by odd numbers. This operation is denoted by $(\downarrow 2)$. The two parts of the signal are transmitted separately, and an estimate of the original signal is produced by reassembling them.

Prior to the reassembly, zeros are interpolated between the elements of the component signals to replace the discarded elements. This operation is described as upsampling, and it is denoted by $(\uparrow 2)$. Then, the upsampled sequences are passed though separate smoothing filters F_1 and F_0, designed to replace the interpolated zeros by estimates of the missing values. Finally, the two signals are added together.

Let the input signal be denoted by $x(t)$ and its z-transform by $x(z)$. Here, z is generally taken to be the complex exponential $e^{-i\omega}$. In that case, we can denote $x(e^{-i\omega})$ more economically by $x(\omega)$. However, by using $x(z)$, we can achieve a greater generality at the same time as easing the burden of notation.

The path taken by the signal through the highpass branch of the network may be denoted by

$$(1) \qquad x(z) \longrightarrow H_1(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow F_1(z) \longrightarrow y_1(z),$$

whereas the path taken through the lowpass branch may be denoted by

$$(2) \qquad x(z) \longrightarrow H_0(z) \longrightarrow (\downarrow 2) \longrightarrow \simeq \longrightarrow (\uparrow 2) \longrightarrow F_0(z) \longrightarrow y_0(z).$$

The symbol $\simeq$ denotes the storage and transmission of the signals. The output signal, formed by merging the two branches, is $y(t) = y_0(t) + y_1(t)$.

Our immediate objective is to find the z transform of the reconstituted signal $y(t)$. Consider any signal $p(t)$ that has been subject to the processes of downsampling and upsampling to produce the sequence $q(t) = \{p(t \downarrow 2) \uparrow 2\}$. Let $p(t) \longleftrightarrow p(\omega)$, which is to say that $p(\omega)$ is the Fourier transform of $p(t)$. Then, it is well now that $p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(\omega/2) + p(\pi + \omega/2)$. Since $e^{\pm\pi} = -1$ and $e^{-(\pi+\omega/2)} = -e^{-\omega/2}$, we can also write

$$(3) \qquad p(t \downarrow 2) \longleftrightarrow \frac{1}{2}\{p(z^{1/2}) + p(-z^{1/2})\}.$$

Next there is a process of upsampling which doubles the value of the frequency argument. This gives $\{p(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(\omega) + p(\pi + \omega)\}$, and we can also write

$$(4) \qquad \{p(t \downarrow 2) \uparrow 2\} \longleftrightarrow \frac{1}{2}\{p(z) + p(-z)\}.$$

It follows from this that the signals that emerge from the two branches of the network are given by

$$
\begin{aligned}
y_1(z) &= \frac{1}{2}F_1(z)\{H_1(z)x(z) + H_1(-z)x(-z)\}, \\
y_0(z) &= \frac{1}{2}F_0(z)\{H_0(z)x(z) + H_0(-z)x(-z)\}.
\end{aligned} \tag{5}
$$

combining the two signals gives

$$
\begin{aligned}
y(z) = \frac{1}{2}&\{F_0(z)H_0(-z) + F_1(z)H_1(-z)\}x(-z), \\
&+ \frac{1}{2}\{F_0(z)H_0(z) + F_1(z)H_1(z)\}x(z).
\end{aligned} \tag{6}
$$

The term in $x(-z)$ is due to aliasing and, by setting

$$
F_0(z) = H_1(-z), \qquad F_1(z) = -H_0(-z), \tag{7}
$$

we can eliminate it to give

$$
\begin{aligned}
y(z) &= \frac{1}{2}\{H_0(z)F_0(z) - F_0(-z)H_0(-z)\}x(z) \\
&= \frac{1}{2}\{P_0(z) - P_0(-z)\}x(z).
\end{aligned} \tag{8}
$$

Having eliminated the aliasing effect, we should like to ensure that the process of two-channel filtering leads to an output signal that is just a delayed version on the input signal. Since z is a one-period lag operator, the condition we seek to impose is that

$$
P_0(z) - P_0(-z) = 2z^{-\ell}, \tag{9}
$$

where ℓ is some integer. Note that the coefficients of the polynomial $P_0(z) - P_0(-z)$ that are associated with even powers of z must be zero-valued, since the terms of $P_0(z)$ and $P_0(-z)$ associated with even powers are the same. Therefore, ℓ is an odd number, and the condition asserts that all coefficients of $P_0(z)$ must the zeros other than the ℓth coefficient, which must be unity.

We can imagine that the ℓth coefficient is the central coefficient of the polynomial $P(z)$. In that case, it is convenient to re-centre the polynomial so that this coefficient is associated with z^0. Therefore, we define $P(z) = z^{\ell}P_0(z)$. Then $P(-z) = (-z)^{\ell}P_0(-z) = -(z^{\ell})P_0(-z)$, where the final equality follows because ℓ is odd. It follows that the condition of (9) can be rewritten as

$$
P(z) + P(-z) = 2. \tag{10}
$$

This has two evident meanings. First, it means that the coefficients of $P(z)$ associated with even powers of z must be zero apart from the coefficient associated with z^0, which is unity. Second, if $\frac{1}{2}P(z)$ corresponds to a lowpass filter, then $\frac{1}{2}P(-z)$ is the complementary high pass filter.

Example. Consider

$$
(11)\qquad
\begin{aligned}
&\text{(i)} && H_0(z) = p_0 + p_1 z + p_2 z^2 + p_3 z^3,\\
&\text{(ii)} && H_1(z) = p_3 - p_2 z + p_1 z^2 - p_0 z^3,\\
&\text{(iii)} && F_0(z) = p_3 + p_2 z + p_1 z^2 + p_0 z^3 = H_1(-z) = z^3 H_0(z^{-1}),\\
&\text{(iv)} && F_1(z) = -p_0 + p_1 z - p_2 z^2 + p_3 z^3 = -H_0(-z) = z^3 H_1(z^{-1}).
\end{aligned}
$$

We observe that the conditions of (7), which eliminate the aliasing, are fulfilled. An additional feature is the relationships between the coefficients of $H_0(z)$ and $H_1(z)$ and those of $F_0(z)$ and $F_1(z)$. It will be see that the inner products of the respective sequences are zero, which implies that they are mutually orthogonal. Observe that $H_0(z)F_0(z) = z^3 H_0(z)H_0(z^{-1}) = P_0(z)$, and, therefore, we can describe $P(z) = z^{-3}P_0(z)$ as an energy function. Finding the filters is essentially a matter of factorising the energy function.