THE PARTIAL FRACTION DECOMPOSTION OF AN AUTOCOVARIANCE GENERATING FUNCTION

Consider the following decomposition

$$y(t) = \frac{1-\gamma L}{(1-\alpha L)(1-\beta L)}\varepsilon(t) = \frac{A}{1-\alpha L}a(t) + \frac{B}{1-\beta L}b(t), \tag{1}$$

where $\varepsilon(t)$ is a white-noise process with $V\{\varepsilon(t)\} = \sigma^2$ and $a(t), b(t)$ are mutually independent white-noise processes with unit variance. For convenience, we shall also set $\sigma^2 = 1$. We can equate the autocovariance generating function of $y(t)$ with the sum of the autocovariance generating functions of the processes on the RHS:

$$\begin{aligned}\frac{(1-\gamma z)(1-\gamma z^{-1})}{(1-\alpha z)(1-\beta z)(1-\beta z^{-1})(1-\alpha z^{-1})}\sigma_\varepsilon^2 \\ = \frac{A^2}{(1-\alpha z)(1-\alpha z^{-1})} + \frac{B^2}{(1-\beta z)(1-\beta z^{-1})},\end{aligned} \tag{2}$$

which also gives

$$(1-\gamma z)(1-\gamma z^{-1}) = A^2(1-\beta z)(1-\beta z^{-1}) + B^2(1-\alpha z)(1-\alpha z^{-1}), \tag{3}$$

or

$$(1+\gamma^2) - \gamma(z+z^{-1}) = A^2(1+\beta^2) + B^2(1+\alpha^2) - (A^2\beta + B^2\alpha)(z+z^{-1}). \tag{4}$$

This provides two equations in two unknowns

$$\begin{bmatrix} 1+\beta^2 & 1+\alpha^2 \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} A^2 \\ B^2 \end{bmatrix} = \begin{bmatrix} 1+\gamma^2 \\ \gamma \end{bmatrix}. \tag{5}$$

The solution is

$$\begin{bmatrix} A^2 \\ B^2 \end{bmatrix} = \frac{1}{\alpha(1+\beta^2) - \beta(1+\alpha^2)}\begin{bmatrix} \alpha(1+\gamma^2) - \gamma(1+\alpha^2) \\ \gamma(1+\beta^2) - \beta(1+\gamma^2) \end{bmatrix}. \tag{6}$$

For an alternative derivation, consider multiplying both sides of (2) by $(1-\alpha z)(1-\alpha z^{-1})$. We get

$$\frac{(1-\gamma z)(1-\gamma z^{-1})}{(1-\beta z)(1-\beta z^{-1})}\sigma_\varepsilon^2 = A^2 + B^2\frac{(1-\alpha z)(1-\alpha z^{-1})}{(1-\beta z)(1-\beta z^{-1})}. \tag{7}$$

Then, setting $z = \alpha$ or $z = \alpha^{-1}$ gives

$$A^2 = \sigma_\varepsilon^2\frac{(1-\gamma\alpha)(\alpha-\gamma)}{(1-\beta\alpha)(\alpha-\beta)}, \tag{8}$$

which agrees with the previous result.

Observe that the solutions to the equation

$$\frac{(1-\gamma z)}{(1-\alpha z)(1-\beta z)} = \frac{A^2}{1-\alpha z} + \frac{B^2}{1-\beta z}, \tag{9}$$

are

$$A = \frac{\alpha - \gamma}{\alpha - \beta} \quad \text{and} \quad B = \frac{\beta - \gamma}{\beta - \alpha}. \tag{10}$$

These agree with above in the case where $\alpha = 1$, $\beta = 1$; and they can certainly be used in finding the solutions of A^2, B^2 in the general case. It is also noteworthy that the solutions to a seemingly quadratic problem are obtained by solving some linear equations.

The Decomposition of a Drifting Seasonal Model

In model above, the MA order of the numerator is less than the AR order of the denominator, which is to say that the transfer function, which transforms the white noise process $\varepsilon(t)$ into the output process $y(t)$, is a proper rational function of the lag operator. In some cases, the numerator order equals or exceeds that of the denominator order. To pursue a partial fraction decomposition, it is necessary first to divide the numerator by the denominator. Then the decomposition is effected with the remainder.

Consider, for example, the seasonal model

$$(I - L^s)y(t) = (1 - \theta L^s)\varepsilon(t), \tag{11}$$

wherein $(I - L^s) = (I - L)(1 + L + \cdots + L^{s-1}) = (I - L)S(L)$ is the seasonal difference operator that delivers the difference $(I - L^s)y(t) = y(t) - y(t - s)$ between the current value and the value from the previous year. The decomposition of the autocovariance generating function of $y(t)$ is

$$\frac{(1-\theta z^s)(1-\theta z^{-s})}{(1-z^s)(1-z^{-s})} = \frac{A(z)}{(1-z)(1-z^{-1})} + \frac{B(z)}{S(z)S(z^{-1})} + \theta, \tag{12}$$

where θ is the quotient from the division of $(1-\theta z^s)(1-\theta z^{-s}) = (1+\theta) - \theta(z^s + z^{-s})$ by $(1-z^s)(1-z^{-s}) = 2 - (z^s + z^{-s})$. Using

$$\frac{(1-\theta z^s)(1-\theta z^{-s})}{(1-z^s)(1-z^{-s})} - \theta = \frac{(1-\theta)^2}{(1-z^s)(1-z^{-s})}, \tag{13}$$

we can rewrite equation (12) as

$$\frac{(1-\theta)^2}{(1-z)S(z)S(z^{-1})(1-z^{-1})} = \frac{A(z)}{(1-z)(1-z^{-1})} + \frac{B(z)}{S(z)S(z^{-1})}, \tag{14}$$

Multiplying both sides by $(1-z)(1-z^{-1})$ gives

$$\frac{(1-\theta)^2}{S(z)S(z^{-1})} = A(z) + B(z)\frac{(1-z)(1-z^{-1})}{S(z)S(z^{-1})}, \tag{15}$$

whence, on setting $z = 1$ we get

$$A(z) = \frac{(1-\theta)^2}{s^2}. \tag{16}$$

Substituting this in (14) gives

$$\begin{aligned} \frac{(1-\theta)^2}{S(z)S(z^{-1})} - A(z) &= \frac{(1-\theta)^2\{1 - S(z)S(z^{-1})/s^2\}}{S(z)S(z^{-1})} \\ &= B(z)\frac{(1-z)(1-z^{-1})}{S(z)S(z^{-1})}, \end{aligned} \tag{17}$$

from which

$$B(z) = \frac{(1-\theta)^2\{1 - S(z)S(z^{-1})/s^2\}}{(1-z)(1-z^{-1})}. \tag{18}$$