FORECASTING WITH THE IMA(1,1) MODEL

Consider the following integrated first-order moving average or IMA(1,1) model:

$$y(t) = y(t-1) + \varepsilon(t) - \theta\varepsilon(t-1). \tag{1}$$

This can also be written as

$$(1-L)y(t) = (1-\theta L)\varepsilon(t). \tag{2}$$

We can readily express the sequence $\varepsilon(t)$ in terms of past and present values of $y(t)$:

$$\begin{aligned}\varepsilon(t) = \frac{1-L}{1-\theta L}y(t) &= (1-L)\{y(t) + \theta y(t-1) + \theta^2 y(t-2) + \cdots\}\\ &= \{y(t) + (\theta-1)y(t-1) + \theta(\theta-1)y(t-2) + \cdots\}.\end{aligned} \tag{3}$$

We hesitate to use a similar manipulation to express $y(t)$ in terms of $\varepsilon(t)$. This would entail the series expansion $(1-L)^{-1}\varepsilon(t) = \{\varepsilon(t) + \varepsilon(t-1) + \cdots\}$ which comprises infinite sums of the elements of the stationary white-noise sequence $\varepsilon(t)$. There can be no presumption, in general, that such sums will be finite-valued.

Imagine, instead, that $y(t)$ is finite-valued at some specified time, and consider advancing the time successively. Advancing it by one period gives

$$y(t+1) = y(t) + \varepsilon(t+1) - \theta\varepsilon(t). \tag{4}$$

Advancing it again gives

$$\begin{aligned}y(t+2) &= y(t+1) + \varepsilon(t+2) - \theta\varepsilon(t+1)\\ &= \{y(t) - \theta\varepsilon(t)\} + (1-\theta)\varepsilon(t+1) + \varepsilon(t+2),\end{aligned} \tag{5}$$

and advancing it successively through h periods gives

$$\begin{aligned}y(t+h) = \{&y(t) - \theta\varepsilon(t)\}\\ &+ (1-\theta)\{\varepsilon(t+1) + \cdots + \varepsilon(t+h-1)\} + \varepsilon(t+h).\end{aligned} \tag{6}$$

Given a finite starting value for the sequence $y(t)$, this can be used to express all subsequent values in term of an accumulation of the elements of to $\varepsilon(t)$.

The forecast of the values for h steps ahead that is made at time t is obtained by finding the expectation of y_{t+h} given the information available at time t. This information consists equally of elements of $\mathcal{I}_t^\varepsilon = \{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$ or the elements $\mathcal{I}_t^y = \{y_t, y_{t-1}, \ldots\}$. Given the value of the parameter θ, there is a complete equivalence between the two sets in the sense that from either one we can find the other.

The expectation of y_{t+1} conditional upon the information available at time t, which is obtained by taking conditional expectations in equation (4), is

$$\begin{aligned}E(y_{t+1}|\mathcal{I}_t) &= E(y_t|\mathcal{I}_t) + E(\varepsilon_{t+1}|\mathcal{I}_t) - \theta E(\varepsilon_t|\mathcal{I}_t)\\ &= y_t - \theta\varepsilon_t.\end{aligned} \tag{7}$$

The corresponding forecast function may be denoted by

$$\hat{y}(t+1|t) = y(t) - \theta\varepsilon(t). \tag{8}$$

Taking expectations in equation (6) gives the same result, which implies that the forecast is the same for all future values:

$$\hat{y}(t+h|t) = \hat{y}(t+1|t). \tag{9}$$

The forecast can be expressed in terms of the observed values of $y(t)$. Substituting equation (3) into (8) gives

$$\begin{aligned} \hat{y}(t+1|t) &= \{(1-\theta)y(t) + \theta(1-\theta)y(t-1) + \theta^2(1-\theta)y(t-2) + \cdots\} \\ &= \frac{1-\theta}{1-\theta L}y(t) \end{aligned} \tag{10}$$

Moreover, Taking (7) from (5) show the forecast error h periods ahead is

(11)

$$\begin{aligned} y(t+h) - \hat{y}(t+h|t) &= \{(1-\theta)\varepsilon(t+1) + \cdots + (1-\theta)\varepsilon(t+h-1)\} + \varepsilon(t+h) \\ &= \{1 + (1-\theta)(L + L^2 + \cdots + L^{h-1})\}\varepsilon(t+h). \end{aligned}$$

The practical means of generating the one-step ahead forecasts of $y(t)$ is via the so-called prediction-error algorithm. Taking $\hat{y}(t|t-1) = y(t-1) - \theta\varepsilon(t-1)$ from $y(t) = y(t-1) + \varepsilon(t) - \theta\varepsilon(t-1)$ indicates that

$$\varepsilon(t) = \hat{y}(t|t-1) - y(t). \tag{12}$$

This is the one-step-ahead predition error. When (12) is used in (8), we get

$$\hat{y}(t+1|t) = y(t) + \theta\{y(t) - \hat{y}(t|t-1)\} \tag{13}$$