Up-sampling and Down-sampling

In this note, we shall consider the effects of up-sampling and down-sampling by factors of 2. We shall treat both the case of an infinite sample and the case of a finite sample.

In the case of a finite sample, the frequency-domain representation and the time-domain representation are connected via the discrete Fourier transform. Let the sample be denoted by $x_t; t = 0, 1, \ldots, T-1$ and let the ordinates of its Fourier transform be denoted by $\xi_j; j = 0, 1, \ldots, T-1$. Then, we have

$$x_t = \frac{1}{T}\sum_{j=0}^{T-1} \xi_j \exp\{\mathrm{i}\omega_1 tj\} \quad \text{and} \tag{1}$$

$$\xi_j = \sum_{t=0}^{T-1} x_t \exp\{-\mathrm{i}\omega_1 jt\}, \quad \text{where} \quad \omega_1 = \frac{2\pi}{T}. \tag{2}$$

We can envisage the (inverse) Fourier transform, represented by (1), in terms of the sequence $\xi_j; j = 0, 1, \ldots, T-1$ disposed, with equal spacing, around the circle of circumference T at angles of $\omega_j = j \times \omega_1 = 2\pi j/T$ from the horizontal.

Down-sampling by a factor of 2 is a matter of selecting alternate sample points. We may assume that T is an even number. Then, the selected sample points are described by

$$\begin{aligned} x_{2t} &= \frac{1}{T}\sum_{j=0}^{T-1} \xi_j \exp\{\mathrm{i}\omega_1 2tj\} \\ &= \frac{2}{T}\sum_{j=0}^{(T/2)-1} \frac{\xi_j + \xi_{j+(T/2)}}{2} \exp\{\mathrm{i}\omega_2 tj\}, \quad \text{where} \quad \omega_2 = \frac{4\pi}{T}, \end{aligned} \tag{3}$$

where the second equality follow from the fact that

$$\exp\{\mathrm{i}\omega_2 j\} = \exp\{\mathrm{i}\omega_2 [j \bmod (T/2)]\}.$$

To envisage the effect of equation (3), we can imagine wrapping the sequence $\xi_j; j = 0, 1, \ldots, T-1$ twice around a circle of circumference $T/2$ so that its elements fall on $T/2$ points at angles of $\omega_{2j} = 4\pi j/T; j = 1, 2, \ldots, T/2$ from the horizontal. This is one possible visualisation. An alternative visualisation leaves the circumference of the circle unaltered while the length of the sequence is doubled by interpolating zeros between its elements. Then, the zero-valued elements will fall upon the frequency values $\omega_j; j = 1, 3, \ldots, T-1$, which are indexed by odd integers, and, if we select only the pairs of elements that fall on the frequency values indexed by even integers, the effect will be the same.

Now consider a doubly infinite sequence indexed by $t \in \{0, \pm 1, \pm 2, \ldots\}$. Then, the so-called discrete-time Fourier transform replaces the discrete Fourier transform, and, in place of (1) and (2), we have

$$x_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega)\exp\{\mathrm{i}\omega t\}d\omega \quad \text{and} \tag{4}$$

$$\xi(\omega) = \sum_{t=-\infty}^{\infty} x_t \exp\{-\mathrm{i}\omega t\}. \tag{5}$$

When alternate values are selected from the sample, equation (4) becomes

$$\begin{aligned} x(2t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(\omega)e^{\mathrm{i}\omega(2t)}d\omega \\ &= \frac{1}{4\pi}\int_{-2\pi}^{2\pi} \xi(\lambda/2)e^{\mathrm{i}\lambda t}d\lambda \\ &= \frac{1}{4\pi}\left\{\int_{-2\pi}^{-\pi} \xi(\lambda/2)e^{\mathrm{i}\lambda t}d\lambda + \int_{-\pi}^{\pi} \xi(\lambda/2)e^{\mathrm{i}\lambda t}d\lambda + \int_{\pi}^{2\pi} \xi(\lambda/2)e^{\mathrm{i}\lambda t}d\lambda\right\}. \end{aligned} \tag{6}$$

Here we have defined $\lambda = 2\omega$, and we have used the change of variable technique to obtain the first equality.

But the integrand of (6) is a periodic function which completes one cycle in 2π radians. Therefore, in the final expression, the first integral may be translated to the interval $(0, \pi]$, by adding 2π to the argument, whereas the third integral may be translated to the interval $(-\pi, 0]$, by subtracting 2π from the argument. After their translation, the first and the third integrands combine to form the segment of the function $\xi(\pi + \lambda/2)$ that falls in the interval $(-\pi, \pi]$. The consequence is that

$$x(2t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\xi(\lambda/2) + \xi(\pi + \lambda/2)}{2} e^{\mathrm{i}\lambda t}d\lambda. \tag{7}$$

It follows that

$$x(2t) \longleftrightarrow \frac{1}{2}\{\xi(\lambda/2) + \xi(\pi + \lambda/2)\}. \tag{8}$$

The superimposition of the shifted frequency function $\xi(\pi + \lambda/2)$ upon the function $\xi(\lambda/2)$ corresponds to a process of aliasing. To visualise the process, we can imagine a single cycle of the original function $\xi(\omega)$ over the interval $[-\pi, \pi]$. The dilated function $\xi(\lambda/2)$ manifests a single cycle over the interval $[-2\pi, 2\pi]$. By wrapping this segment twice around the circle defined by $\exp\{-i\lambda\}$ with $\lambda \in [-\pi, \pi]$, we obtain the aliased function $\{\xi(\lambda/2) + \xi(\pi + \lambda/2)\}$.

The problem of aliasing can be overcome by applying an anti-aliasing filter to the data prior to down-sampling it. The ideal anti-aliasing filter, in the case of down-sampling by a factor of 2, is the so-called Shannon lowpass half-band filter which is defined by the following sinc function:

$$\phi(t) = \frac{\sin(\pi t/2)}{\pi t}, \tag{9}$$

In case it is the upper half of the frequency spectrum that one wishes to preserve, a half-band highpass filter can be used which is given by

$$\psi(t) = \frac{\cos(\pi t)\sin(\pi t/2)}{\pi t}. \tag{10}$$

In the case of a finite sample, the Fourier transforms of the two filters are given, respectively, by the following periodic square-wave or boxcar functions defined on the frequency points $\omega_j = 2\pi j/T$:

$$\begin{aligned} \phi(\omega_j) &= \begin{cases} 1, & \text{if } \omega_j \in (\pi[2k-1]/2, \pi[2k+1]/2), \quad k \text{ even}, \\ 1/2, & \text{if } \omega_j = \pi k/2, \quad k \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \\ \psi(\omega_j) &= \begin{cases} 1, & \text{if } \omega_j \in (\pi[2k-1]/2, \pi[2k+1]/2), \quad k \text{ odd}, \\ 1/2, & \text{if } \omega_j = \pi k/2, \quad k \text{ odd} \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{11}$$

Applying the anti-aliasing filters to the data involves a process of convolution. To appy the filters to the corresponding spectral ordinates only involves multiplying them by the appropriate box-car elements. This requires much less computation.

In the case of a finite sample of size T, the spectral ordinates that are obtained by down-sampling without anti-aliasing are the values

$$\frac{\xi_j + \xi_{j+(T/2)}}{2}; j = 0, 1, \ldots, (T/2) - 1,$$

which are to be found in equation (3). With anti-aliasing, the ordinates become

$$\frac{\xi_0 + \xi_{T/2}}{4}, \frac{\xi_1}{2}, \ldots, \frac{\xi_{(T/2)-1}}{2}.$$

Now let us consider the process of up-sampling by a factor of 2 on the basis of the finite sample $x_t; t = 0, 1, \ldots, T-1$. This is a matter of interpolating zeros between each of the sample elements to create a sample $\hat{x}_s; s = 0, 1, \ldots, 2T-1$ of length $2T$, such that $\hat{x}_{2t} = x_t$ and $\hat{x}_{2t+1} = 0$.

The ordinates $\hat{\xi}_k; k = 0, 1, \ldots, 2T-1$ of the Fourier transform of this interpolated sample are given by

$$\begin{aligned}\hat{\xi}_k &= \sum_{s=0}^{2T-1} \hat{x}_s \exp\{-\mathrm{i}\omega_{1/2} ks\} \\ &= \sum_{t=0}^{T-1} x_t \exp\{-\mathrm{i}\omega_1 kt\}, \quad \text{where} \quad \omega_{1/2} = \frac{\pi}{T}.\end{aligned} \tag{12}$$

But $\exp\{-\mathrm{i}\omega_1 kt\}$ is a periodic function of the index k with a period of T, so it follow that $\hat{\xi}_k = \hat{\xi}_{k \bmod T}$. Since $k = 0, 1, \ldots, 2T-1$, the sequence of frequency-domain coefficients $\hat{\xi}_k; k = 0, 1, \ldots, 2T-1$ will manifest two cycles within the range of the index, which indexes a set of frequency values equally spaced in the interval $[0, 2\pi]$. The original sequence $\xi_j; j = 0, 1, \ldots, T-1$ manifests only single cycle within the same frequency range. Moreover, the two sequences share the same set of ordinates. The process of creating two cycles from one, which is a matter of doubling the value of the argument of the frequency domain function, is described as imaging.

For a visualisation of this result, we may think of the original frequency-domain ordinates $\xi_j = \xi_{kT+j}; j, k \in \{0, \pm 1, \pm 2, \ldots\}$ as the elements of a periodic function. The process of interpolation doubles the number of frequency values that are equally spaced around the circle. Two cycles of the periodic sequence are now required in order to encompass the circle.

Now consider the matter of up-sampling a doubly infinite data sequence. It is clear that

$$\hat{\xi}(\omega) = \sum_{s=-\infty}^{\infty} \hat{x}_s \exp\{-\mathrm{i}\omega s\} = \sum_{t=-\infty}^{\infty} x_t \exp\{-\mathrm{i}2\omega t\} = \xi(2\omega). \tag{13}$$

This is just a matter of doubling the frequency argument.

To overcome the problem of imaging, it is necessary to apply a half-band filter to the interpolated data sequence. This entails the convolution of the data values with coefficients of the lowpass half-band Shannon filter. The process consists of replacing each of the data points by a sinc function, scaled by the value of the point, and adding the results. The effect will be to replace the interpolated zeros by appropriate values, without affecting the original sample values, which are on alternate points.

To understand this result, we may note that the sinc functions have zeros on alternate indices. That is to say, we find that $\psi(2t) = \phi(2t) = 0$ for $t \neq 0$. Therefore, if any element of a sequence of doubly-spaced sample points is replaced by a sinc function, whose central value is scaled so that it equals that of the point it replaces, then the only effect will be to interpolate values on the alternate points where formerly there were zeros. The sample points are not affected.