TRANSFER FUNCTIONS

Consider a simple dynamic model of the form

$$y(t) = \phi y(t-1) + x(t)\beta + \varepsilon(t). \tag{1}$$

With the use of the lag operator, we can rewrite this as

$$(1-\phi L)y(t) = \beta x(t) + \varepsilon(t) \tag{2}$$

or, equivalently, as

$$y(t) = \frac{\beta}{1-\phi L}x(t) + \frac{1}{1-\phi L}\varepsilon(t). \tag{3}$$

The latter is the so-called rational transfer-function form of the equation. We can replace the operator L within the transfer functions or filters associated with the signal sequence $x(t)$ and disturbance sequence $\varepsilon(t)$ by a complex number z. Then, for the transfer function associated with the signal, we get

$$\frac{\beta}{1-\phi z} = \beta\bigl\{1 + \phi z + \phi^2 z^2 + \cdots\bigr\}, \tag{4}$$

where the RHS comes from a familiar power-series expansion.

The sequence $\{\beta, \beta\phi, \beta\phi^2, \ldots\}$ of the coefficients of the expansion constitutes the impulse response of the transfer function. That is to to say, if we imagine that, on the input side, the signal is a unit-impulse sequence of the form

$$x(t) = \{\ldots, 0, 1, 0, 0, \ldots\}, \tag{5}$$

which has zero values at all but one instant, then its mapping through the transfer function would result in an output sequence of

$$r(t) = \{\ldots, 0, \beta, \beta\phi, \beta\phi^2, \ldots\}. \tag{6}$$

Another important concept is the step response of the filter. We may imagine that the input sequence is zero-valued up to a point in time when it assumes a constant unit value:

$$x(t) = \{\ldots, 0, 1, 1, 1, \ldots\}. \tag{7}$$

The mapping of this sequence through the transfer function would result in an output sequence of

$$s(t) = \{\ldots, 0, \beta, \beta + \beta\phi, \beta + \beta\phi + \beta\phi^2, \ldots\} \tag{8}$$

whose elements, from the point when the step occurs in $x(t)$, are simply the partial sums of the impulse-response sequence.

This sequence of partial sums $\{\beta, \beta + \beta\phi, \beta + \beta\phi + \beta\phi^2, \ldots\}$ is described as the step response. Given that $|\phi| < 1$, the step response converges to a value

$$\gamma = \frac{\beta}{1-\phi} \tag{9}$$

which is described as the steady-state gain or the long-term multiplier of the transfer function.

These various concepts apply to models of any order. Consider the equation

$$\alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t), \tag{10}$$

where

$$\begin{aligned} \alpha(L) &= 1 + \alpha_1 L + \cdots + \alpha_p L^p \\ &= 1 - \phi_1 L - \cdots - \phi_p L^p, \\ \beta(L) &= 1 + \beta_1 L + \cdots + \beta_k L^k \end{aligned} \tag{11}$$

are polynomials of the lag operator. The transfer-function form of the model is simply

$$y(t) = \frac{\beta(L)}{\alpha(L)}x(t) + \frac{1}{\alpha(L)}\varepsilon(t), \tag{12}$$

The rational function associated with $x(t)$ has a series expansion

$$\begin{aligned} \frac{\beta(z)}{\alpha(z)} &= \omega(z) \\ &= \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}; \end{aligned} \tag{13}$$

and the sequence of the coefficients of this expansion constitutes the impulse-response function. The partial sums of the coefficients constitute the step-response function. The gain of the transfer function is defined by

$$\gamma = \frac{\beta(1)}{\alpha(1)} = \frac{\beta_0 + \beta_1 + \cdots + \beta_k}{1 + \alpha_1 + \cdots + \alpha_p}. \tag{14}$$

The method of finding the coefficients of the series expansion of the transfer function in the general case can be illustrated by the second-order case:

$$\frac{\beta_0 + \beta_1 z}{1 - \phi_1 z - \phi_2 z^2} = \{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}. \tag{15}$$

We rewrite this equation as

$$\beta_0 + \beta_1 z = \{1 - \phi_1 z - \phi_2 z^2\}\{\omega_0 + \omega_1 z + \omega_2 z^2 + \cdots\}. \tag{16}$$

Then, by performing the multiplication on the RHS, and by equating the coefficients of the same powers of z on the two sides of the equation, we find that

$$(17)\qquad \begin{aligned} \beta_0 &= \omega_0, & \omega_0 &= \beta_0,\\ \beta_1 &= \omega_1 - \phi_1\omega_0, & \omega_1 &= \beta_1 + \phi_1\omega_0,\\ 0 &= \omega_2 - \phi_1\omega_1 - \phi_2\omega_0, & \omega_2 &= \phi_1\omega_1 + \phi_2\omega_0,\\ &\vdots & &\vdots\\ 0 &= \omega_n - \phi_1\omega_{n-1} - \phi_2\omega_{n-2}, & \omega_n &= \phi_1\omega_{n-1} + \phi_2\omega_{n-2}. \end{aligned}$$

By examining this scheme, we are able to distinguish between the different roles which are played by the numerator parameters β_0, β_1 and the denominator parameters ϕ_1, ϕ_2. The parameters of the numerator serve as initial conditions for the process which generates the impulse response. The denominator parameters determine the dynamic nature of the impulse response.

Consider the case where the impulse response takes the form a damped sinusoid. This case arises when the roots of the equation $\alpha(z) = 1 - \phi z - \phi_2 z^2 = 0$ are a pair of conjugate complex numbers falling outside the unit circle—as they are bound to do if the response is to be a damped one. Then the parameters β_0 and β_1 are jointly responsible for the initial amplitude and for the phase of the sinusoid. The phase is the time lag which displaces the peak of the sinusoid so that it occurs after the starting time $t = 0$ of the response, which is where the peak of an undisplaced cosine response would occur.

The parameters ϕ_1 and ϕ_2, on the other hand, serve to determine the period of the sinusoidal fluctuations and the degree of damping, which is the rate at which the impulse response converges to zero.

It seems that all four parameters ought to be present in a model which aims at capturing any of the dynamic responses of which a second-order system is capable. To omit one of the numerator parameters of the model would be a mistake unless, for example, there is good reason to assume that the impulse response attains its maximum value at the starting time $t = 0$. We are rarely in the position to make such an assumption.