HYPOTHESES CONCERNING SUBSETS OF THE REGRESSION COEFFICIENTS

Consider a set of linear restrictions on the vector β of a classical linear regression model $N(y; X\beta, \sigma^2 I)$ which take the form of

$$(1)\qquad R\beta = r,$$

where R is a matrix of order $j \times k$ and of rank j, which is to say that the j restrictions are independent of each other and are fewer in number than the parameters within β. We know that the ordinary least-squares estimator of β is a normally distributed vector $\hat{\beta} \sim N\{\beta, \sigma^2(X'X)^{-1}\}$. It follow that

$$(2)\qquad R\hat{\beta} \sim N\big\{R\beta = r, \sigma^2 R(X'X)^{-1}R'\big\};$$

and, from this, we can immediately infer that

$$(3)\qquad \frac{(R\hat{\beta} - r)'\big\{R(X'X)^{-1}R'\big\}^{-1}(R\hat{\beta} - r)}{\sigma^2} \sim \chi^2(j).$$

We have already established the result that

$$(4)\qquad \frac{(T-k)\hat{\sigma}^2}{\sigma^2} = \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{\sigma^2} \sim \chi^2(T-k)$$

is a chi-square variate which is statistically independent of the chi-square variate

$$(5)\qquad \frac{(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)}{\sigma^2} \sim \chi^2(k)$$

derived from the estimator of the regression parameters. The variate of (4) must also be independent of the chi-square of (3); and it is straightforward to deduce that

$$(6)\qquad \begin{aligned} F &= \left\{ \frac{(R\hat{\beta} - r)'\big\{R(X'X)^{-1}R'\big\}^{-1}(R\hat{\beta} - r)}{j} \bigg/ \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T-k} \right\} \\ &= \frac{(R\hat{\beta} - r)'\big\{R(X'X)^{-1}R'\big\}^{-1}(R\hat{\beta} - r)}{\hat{\sigma}^2 j} \sim F(j, T-k), \end{aligned}$$

which is to say that the ratio of the two independent chi-square variates, divided by their respective degrees of freedom, is an F statistic. This statistic, which embodies only know and observable quantities, can be used in testing the validity of the hypothesised restrictions $R\beta = r$.

A specialisation of the statistic under (6) can also be used in testing an hypothesis concerning a subset of the elements of the vector β. Let $\beta' = [\beta_1', \beta_2']'$. Then the condition that the subvector β_1 assumes the value of β_1^* can be expressed via the equation

$$(7)\qquad [I_{k_1}, 0]\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \beta_1^*.$$

This can be construed as a case of the equation $R\beta = r$ where $R = [I_{k_1}, 0]$ and $r = \beta_1^*$.

In order to discover the specialised form of the requisite test statistic, let us consider the following partitioned form of an inverse matrix:

$$\begin{aligned}(X'X)^{-1} &= \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \{X_1'(I-P_2)X_1\}^{-1} & -\{X_1'(I-P_2)X_1\}^{-1}X_1'X_2(X_2'X_2)^{-1} \\ -\{X_2'(I-P_1)X_2\}^{-1}X_2'X_1(X_1'X_1)^{-1} & \{X_2'(I-P_1)X_2\}^{-1} \end{bmatrix},\end{aligned} \tag{8}$$

Then, with $R = [I, 0]$, we find that

$$R(X'X)^{-1}R' = \left\{X_1'(I-P_2)X_1\right\}^{-1} \tag{9}$$

It follows in a straightforward manner that the specialised form of the F statistic of (6) is

$$\begin{aligned}F &= \left\{ \frac{(\hat{\beta}_1 - \beta_1^*)'\left\{X_1'(I-P_2)X_1\right\}(\hat{\beta}_1 - \beta_1^*)}{k_1} \Big/ \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T-k} \right\} \\ &= \frac{(\hat{\beta}_1 - \beta_1^*)'\left\{X_1'(I-P_2)X_1'\right\}(\hat{\beta}_1 - \beta_1^*)}{\hat{\sigma}^2 k_1} \sim F(k_1, T-k).\end{aligned} \tag{10}$$