STRATEGIES OF HYPOTHESIS TESTING

Joint Tests and Separate Tests

In the case where there are several hypothesis to be tested, we have to decide whether to test jointly using an $F(j, T-k)$ statistic or separately using several $t(T-K)$ statistics. The critical region of the F test will differ from the critical region implied by the t tests. If we choose to test the hypotheses separately using t tests, then we may have difficulty in assessing the size (i.e. the probability of committing a Type I error) of the implicit joint test.

Take, for example, the case of a regression equation with two explanatory variables. If we test the hypotheses that $\beta_1 = 0$ and $\beta_2 = 0$ jointly, then a Type I error, which is a case of rejecting the hypothesis when it is true, occurs when the point $(\hat{\beta}_1, \hat{\beta}_2)$ falls outside an elliptical probability contour. If we test the hypotheses separately, then a Type I error occurs in the implicit joint test when the point $(\beta_1, \hat{\beta}_2)$ falls outside a rectangle whose boundaries are defined by the critical points of the two tests.

Let E_1 be the event of a Type I error in the test of $\beta_1 = 0$, and let E_2 be the event of a Type I error in the test of $\beta_2 = 0$. Then $E = E_1 \cup E_2$ is the event of a Type I error in the implicit joint test of $\beta_1, \beta_2 = 0$. Since

$$P(E) = P(E_1) + P(E_2) - P(E_1 \cap E_2), \tag{1}$$

it follows that

$$P(E) \leq P(E_1) + P(E_2).$$

Therefore, if the significance levels in the separates tests are α_1, α_2 and that of the implicit joint test is α, then $\alpha \leq \alpha_1 + \alpha_2$, which gives an upper bound on the size of the implicit joint test. If the tests on $\beta_1 = 0$, and $\beta_2 = 0$ are statistically independent, then, of course,

$$P(E) = P(E_1) + P(E_2) - P(E_1)P(E_2), \tag{2}$$

and $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

Statistically independent tests of the hypotheses regarding the regression parameters can arise when the vectors of the regressors are mutually orthogonal. An example occurs in the Fourier analysis of time series when we regress the observations on a set of sine and cosine functions of various periodicities which are chosen in such a way as to make the regressors orthogonal.

Nested Tests

Consider, for example, the polynomial regression

$$y = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \varepsilon. \tag{3}$$

We might begin by testing the hypothesis $H_1 : \alpha_3 = 0$. If this were accepted, then we could proceed to test $H_2 : \alpha_2 = 0$, and, if the latter were accepted, then we should test the final hypothesis $H_3 : \alpha_1 = 0$.

Although these three hypotheses can be tested separately without reference to each other, they naturally form a nested sequence

$$
\begin{aligned}
&H_1 \supset H_2 \supset H_3 \qquad \text{where}\\
&H_1 : \alpha_3 = 0,\\
&H_2 : \alpha_2 = 0|\alpha_3 = 0,\\
&H_3 : \alpha_1 = 0|\alpha_2 = 0, \alpha_3 = 0.
\end{aligned} \tag{4}
$$

It can be shown that, if the final hypotheses is true, then the tests in a nested sequence are statistically independent, at least asymptotically. This fact greatly assists the determination of the size of the implicit joint test.

To understand this result, we may consider a sequence of regressions models

$$
\begin{aligned}
&H_1 : y \sim N(X\beta, \sigma^2 I),\\
&H_2 : y \sim N(Q\gamma, \sigma^2 I),\\
&H_3 : y \sim N(Z\delta, \sigma^2 I), \qquad \text{where}\\
&\qquad \mathcal{M}(Z) \subset \mathcal{M}(Q) \subset \mathcal{M}(X).
\end{aligned} \tag{5}
$$

A simple case of this would be where $X = [W_1, W_2, W_3]$, $Q = [W_1, W_2]$ and $Z = [W_1]$. We define the projectors $P_X = X(X'X)^{-1}X'$, $P_Q = Q(Q'Q)^{-1}Q'$, $P_Z = Z(Z'Z)^{-1}Z'$ which are subject to the conditions that $P_XP_Q = P_Q$, $P_XP_Z = P_Z$ and $P_QP_Z = P_Z$. To test the hypothesis H_2, we should use the F statistic

$$
F = \left\{ \frac{(X\hat{\beta} - Q\hat{\gamma})'(X\hat{\beta} - Q\hat{\gamma})}{k_X - k_Q} \Big/ \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T - k_X} \right\} \sim F(k_X - k_Q, T - k_X). \tag{6}
$$

To test the hypothesis H_3 on the assumption that H_2 is true, we should use the F statistic

$$
F = \left\{ \frac{(Q\hat{\gamma} - Z\hat{\delta})'(Q\hat{\gamma} - Z\hat{\delta})}{k_Q - k_Z} \Big/ \frac{(y - Q\hat{\gamma})'(y - Q\hat{\gamma})}{T - k_Q} \right\} \sim F(k_Q - k_Z, T - k_Q). \tag{7}
$$

It is straightforward to show that the numerators of these two statistics represent independent chi-square variates; The can be written respectively as

$$
\begin{aligned}
&\sigma^{-2}(P_Xy - P_Qy)'(P_Xy - P_Qy) \sim \chi^2(k_X - k_Q),\\
&\sigma^{-2}(P_Qy - P_Zy)'(P_Qy - P_Zy) \sim \chi^2(k_Q - k_Z).
\end{aligned} \tag{8}
$$

Their independence follows from showing that $(P_X - P_Q)y$ and $(P_Q - P_Z)y$ are mutually orthogonal. It is sufficient to show that

$$
\begin{aligned}
(P_X - P_Q)'(P_Q - P_Z) &= P_XP_Q - P_Q - P_XP_Z + P_QP_Z\\
&= P_Q - P_Q - P_Z + P_Q = 0.
\end{aligned} \tag{9}
$$

Given that the hypothesis H_2 is valid, which entails the validity of H_1, it also follows that $(y - X\hat{\beta})'(y - X\hat{\beta})/(T - k_X)$ and $(y - Q\hat{\gamma})'(y - Q\hat{\gamma})/(T - k_Q)$ will tend in probability to the parameter σ^2. It follows that, under H_3, the two F statistics are asymptotically independent. This means that, if α_2 is the significance level of the test of H_2 and if α_3 is the significance level of the test of H_3, then the significance level of the implicit test of $H_2 \cap H_3$ will tend asymptotically to $\alpha_2 + \alpha_3 - \alpha_2\alpha_3$ which, with the usual significance levels, is adequately approximated by $\alpha_2 + \alpha_3$.

When a certain conclusion about the validity of the ulterior hypothesis has been reached at the end of a nested sequence of tests, then there is always the option of testing the implicit hypothesis directly in a single test. Thus a direct test of H_3 can be conducted using the following statistic:

$$(10) \qquad F = \left\{ \frac{(X\hat{\beta} - Z\hat{\delta})'(X\hat{\beta} - Z\hat{\delta})}{k_X - k_Z} \bigg/ \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{T - k_X} \right\} \sim F(k_Q - k_Z, T - k_Q).$$

Given that the the critical region of the direct test differs from that of the implicit joint test—even when the tests are of the same size—there is always a possibility that the tests will point to opposite conclusions.