Stochastic Differential Equations

The First-Order Equation

The first-order stochastic differential equation takes the form of

$$(D-\lambda)x(t)=\varepsilon(t) \qquad \text{or} \qquad \frac{dx}{dt}-\lambda x=\varepsilon(t). \tag{1}$$

Multiplying throughout by the factor $\exp\{-\lambda t\}$ gives

$$e^{-\lambda t}Dx(t)-\lambda e^{-\lambda t}x(t)=D\{x(t)e^{-\lambda t}\}=e^{-\lambda t}\varepsilon(t), \tag{2}$$

where the first equality follows from the product rule of differentiation. Integrating $D\{x(t)e^{-\lambda t}\}=e^{-\lambda t}\varepsilon(t)$ gives

$$x(t)e^{-\lambda t}=\int_{-\infty}^{t}e^{-\lambda\tau}\varepsilon(\tau)d\tau \tag{3}$$

or

$$x(t)=e^{\lambda t}\int_{-\infty}^{t}e^{-\lambda\tau}\varepsilon(\tau)d\tau=\int_{-\infty}^{t}e^{\lambda(t-\tau)}\varepsilon(\tau)d\tau \tag{4}$$

If we write $x(t)=(D-\lambda)^{-1}\varepsilon(t)$, then we get the result that

$$x(t)=\frac{1}{D-\lambda}\varepsilon(t)=\int_{-\infty}^{t}e^{\lambda(t-\tau)}\varepsilon(\tau)d\tau. \tag{5}$$

The general solution of a differential equation should normally comprise a particular solution, which represents the effects of the initial conditions. However, given that their effects decay as time elapses and given that, in this case, the integral has no lower limit, no account need be taken of initial conditions.

When the process is observed at the integer time points $\{t=0,\pm1,\pm2,\ldots\}$, it is appropriate to express it as

$$\begin{aligned}x(t)&=e^{\lambda}\int_{0}^{t-1}e^{\lambda(t-1-r)}\varepsilon(r)dr+\int_{t-1}^{t}e^{\lambda(t-r)}\varepsilon(r)dr\\&=e^{\lambda}x(t-1)+\int_{t-1}^{t}e^{\lambda(t-r)}\varepsilon(r)dr.\end{aligned} \tag{6}$$

This gives rise to a discrete-time equation of the form

$$x(t)=\phi x(t-1)+\nu(t), \tag{7}$$

where

$$\phi = e^{\lambda} \qquad \text{and} \qquad \nu(t) = \int_{t-1}^{t} e^{\lambda(t-r)}\varepsilon(r)dr. \tag{8}$$

To calculate the autocovariances of the process, we may take

$$\begin{aligned} \gamma(\tau) &= E\{x(t)x(t+\tau)\} \\ &= \int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{\lambda(t-u)}e^{\lambda(t-v+\tau)}E\{\varepsilon(u)\varepsilon(v)\}dudx \\ &= \sigma_\varepsilon^2 e^{\lambda\tau}\int_{-\infty}^{t}\int_{-\infty}^{t+\tau} e^{\lambda(t-u)}e^{\lambda(t-v)}\delta(u-v)dudv \\ &= \sigma_\varepsilon^2 e^{\lambda\tau}\int_{-\infty}^{t} e^{2\lambda(t-v)}dv \end{aligned} \tag{9}$$

But

$$\int_{-\infty}^{t} e^{2\lambda(t-v)}dv = \int_{0}^{\infty} e^{2\lambda u}du = \frac{1}{2\lambda}, \tag{10}$$

so

$$\gamma(\tau) = \frac{\sigma_\varepsilon^2}{2\lambda}e^{\lambda\tau}. \tag{11}$$

The Second-Order Equation

The solution of second-order equation may be obtained by considering the equation

$$(D^2 + \alpha_1 D + \alpha_2)x(t) = (D-\lambda_1)(D-\lambda_2)x(t) = \varepsilon(t). \tag{12}$$

Using a partial fraction expansion, this can be cast in the form of

$$\begin{aligned} x(t) &= \frac{1}{\lambda_1-\lambda_2}\left\{\frac{1}{D-\lambda_1} - \frac{1}{D-\lambda_2}\right\}\varepsilon(t) \\ &= \int_{-\infty}^{t}\left\{\frac{e^{\lambda_1(t-u)} - e^{\lambda_2(t-u)}}{\lambda_1-\lambda_2}\right\}\varepsilon(u)du \end{aligned} \tag{13}$$

Here, the final equality depends upon the result under (5).