REPARAMETRISATION OF DYNAMIC MODELS

Reparametrisation of a Distributed Lag Model

Consider a distributed-lag model of the form

$$y(t) = \beta_0 x(t) + \beta_1 x(t-1) + \cdots + \beta_k x(t-k) + \varepsilon(t). \tag{1}$$

This can be written is summary notation as

$$y(t) = \beta(L)x(t) + \varepsilon(t), \tag{2}$$

where

$$\beta(L) = \beta_0 + \beta_1 L + \cdots + \beta_k L^k \tag{3}$$

is a polynomial in the lag operator L. We wish to show how this can be reparametrised as an error-correction model.

Define the vectors

$$\beta' = [\beta_0, \beta_1, \ldots, \beta_k] \quad \text{and} \quad x = [x_t, x_{t-1}, \ldots, x_{t-k}]', \tag{4}$$

and let Λ be an arbitrary nonsingular matrix of order $(k+1) \times (k+1)$. Then

$$\beta' x = \{\beta'\Lambda\}\{\Lambda^{-1}x\} = \delta' z, \tag{5}$$

where $\delta' = \beta'\Lambda$ is a vector of alternative parameters and $z = \Lambda^{-1}x$ is a vector of transformed variables.

A simple reparametrisation of this sort can used in forming an error correction model:

$$\begin{aligned}\beta\Lambda &= [\beta_0, \beta_1, \ldots, \beta_{k-1}, \beta_k]\begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 0 \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \\ &= [\kappa, -\delta_1, \ldots, -\delta_{k-1}, -\delta_k]\end{aligned} \tag{6}$$

$$\Lambda^{-1}x = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ x_{t-1} \\ \vdots \\ x_{t-k+1} \\ x_{t-k} \end{bmatrix} = \begin{bmatrix} x_t \\ -\nabla x_t \\ \vdots \\ -\nabla x_{t-k+2} \\ -\nabla x_{t-k+1} \end{bmatrix} \tag{7}$$

These identities enable us to rewrite equation (2) as

$$\begin{aligned} y(t) &= \beta(L)x(t) + \varepsilon \\ &= \beta(1)x(t) + \nabla x(t)\delta(L) + \varepsilon. \end{aligned} \tag{8}$$

where

$$\begin{aligned}\beta(1) &= \beta_0 + \beta_1 + \cdots + \beta_k = \kappa \quad \text{and} \\ \delta(L) &= \delta_1 L + \cdots + \delta_{k-1} L^{k-1} \quad \text{with} \\ \delta_j &= -(\beta_j + \beta_{j+1} + \cdots + \beta_k).\end{aligned} \tag{9}$$

Taking $y(t)$ from both sides of equation (8) gives the error-correction formulation of equation (2):

$$\nabla y(t-1) = \Big\{\kappa x(t) - y(t-1)\Big\} + \nabla\delta(L)x(t) + \varepsilon(t). \tag{10}$$

A Variety of Reparametrisations

There is a variety of reparametrisations which can serve the same essential purpose of expressing the transfer function $\beta(L)x(t)$ in terms of a level of $x(t)$ and the differences of $x(t)$. Consider first the identity

$$\beta(1) = L\beta(1) + \nabla\beta(1). \tag{11}$$

On substituting this into the equation

$$\beta(L) = \beta(1) + \nabla\delta(L), \tag{12}$$

which is from (8), we get

$$\begin{aligned}\beta(L) &= L\beta(1) + \nabla\beta(1) + \nabla\delta(L) \\ &= L\beta(1) + \nabla\delta_1(L).\end{aligned} \tag{13}$$

In particular, it will be observed in reference to (9), that the leading element of $\delta_1(L)$ is

$$\beta(1) = \beta_0. \tag{14}$$

To demonstrate the full range of possibilities, let us consider the identity

$$1 = L^n + \nabla(1 + L + \cdots + L^{n-1}), \tag{15}$$

where n does not exceed the maximim lag in $\beta(L)$. Then

$$\beta(1) = L^n\beta(1) + \nabla\{1 + L + \cdots + L^{n-1}\}\beta(1), \tag{16}$$

and, on defining

$$\delta_n(L) = \delta(L) + \{1 + L + \cdots + L^{n-1}\}\beta(1), \tag{17}$$

we can write

$$\beta(L) = L^n\beta(1) + \nabla\delta_n(L). \tag{18}$$

The leading element of $\delta_n(L)$ continues to be provided by (14).

Reparametrisation of an Autoregressive Distributed Lag Model

Now consider an equation in the form of

$$y(t) = \phi_1 y(t-1) + \cdots + \phi_p y(t-p) + \beta_0 x(t) + \cdots + \beta_k x(t-k) + \varepsilon(t). \tag{19}$$

which can be written is summary notation as

$$\alpha(L)y(t) = \beta(L)x(t) + \varepsilon(t). \tag{20}$$

Using the identity of (13) for $\beta(L)$ and a similar identity for $\alpha(L)$, we may write

$$\beta(L) = \beta(1)L + \delta_1(L)\nabla, \tag{21}$$

$$\begin{aligned}\alpha(L) &= \alpha(1)L + \theta_1(L)\nabla \\ &= L\alpha(1) + \{\nabla - \rho(L)\nabla\}.\end{aligned} \tag{22}$$

Here, equation (22) depends on the condition that $\alpha_0 = 1$, which accounts for the fact that ∇ is found on the RHS in association with a unit coefficient. Substituting (21) and (22) in (20) gives

$$\Big\{\alpha(1)L + \nabla - \rho(L)\nabla\Big\}y(t) = \Big\{\beta(1)L + \nabla\delta_1(L)\Big\}x(t) + \varepsilon(t). \tag{23}$$

This can be rearranged to give

$$\begin{aligned}\nabla y(t) &= \Big\{\beta(1)Lx(t) - \alpha(1)Ly(t)\Big\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t) \\ &= \lambda\Big\{\gamma x(t-1) - y(t-1)\Big\} + \delta_1(L)\nabla x(t) + \rho(L)\nabla y(t) + \varepsilon(t),\end{aligned} \tag{24}$$

where $\lambda = \alpha(1)$ is the so-called adjustment parameter and where $\gamma = \kappa/\lambda = \beta(1)/\alpha(1)$ is the steady-state gain of the rational transfer function $\beta(L)/\alpha(L)$. The term $\gamma x(t-1) - y(t-1)$ is described as the equilibrium error; and the value of the error will tend to zero if a steady state is maintained by $x(t)$ and if there are no distrubances. Equation (24) is the classical form of the error-correction equation.