THE GEOMETRY OF QUADRATIC FORMS

The Circle. Let the coordinates of the points in the Cartersian plane be denoted by (z_1, z_2). Then the equation of a circle of radius r centred on the origin is just

$$z_1^2 + z_2^2 = r^2. \tag{1}$$

This follows immediately from Pythagorus. The so-called parametric equations for the coordinates of the circle are

$$z_1 = r\cos(\omega), \quad \text{and} \quad z_2 = r\sin(\omega). \tag{2}$$

The Ellipse. The equation of an ellipse whose principal axes are aligned with those of the coordinate system in the (y_1, y_2) plane is

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = r^2, \tag{3}$$

On setting $\lambda_1 y_1^2 = z_1^2$ and $\lambda_2 y_2^2 = z_2^2$, we can see that

$$y_1 = \frac{z_1}{\sqrt{\lambda_1}} = \frac{r}{\sqrt{\lambda_1}}\cos(\omega), \quad y_2 = \frac{z_2}{\sqrt{\lambda_2}} = \frac{r}{\sqrt{\lambda_2}}\sin(\omega). \tag{4}$$

We can write equation (3) in matrix notation as

$$r^2 = [\, y_1 \quad y_2 \,] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = z_1^2 + z_2^2. \tag{5}$$

This implies

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{7}$$

The Oblique Ellipse. An oblique ellipse is one whose principal axes are not aligned with those of the coordinate system. Its general equation is

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = r^2; \tag{8}$$

which is subject to the condition that $a_{11}a_{22} - 2a_{12} > 0$. We can write this in matrix notation:

$$
\begin{aligned}
r^2 &= [\,x_1 \quad x_2\,] \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= [\,y_1 \quad y_2\,] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = z_1^2 + z_2^2,
\end{aligned} \tag{9}
$$

where θ is the angle which the pincipcal axis of the ellipse makes with the horizontal. The coefficients of the equation (8) are the elements of the matrix

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta & (\lambda_2 - \lambda_1)\cos\theta\sin\theta \\ (\lambda_2 - \lambda_1)\cos\theta\sin\theta & \lambda_1 \sin^2\theta + \lambda_2\cos^2\theta \end{bmatrix}. \tag{10}
$$

Notice that, if $\lambda_1 = \lambda_2$, which is to say that both axes are rescaled by the same factor, then the equation is that of a circle of radius λ_1, and the rotation of the circle has no effect.

The mapping from the ellipse to the circle is

$$
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1}(x_1\cos\theta - x_2\sin\theta) \\ \sqrt{\lambda_2}(x_1\sin\theta + x_2\cos\theta) \end{bmatrix}, \tag{11}
$$

and the inverse mapping, from the circle to the ellipse, is

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 \\ 0 & 1/\sqrt{\lambda_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}. \tag{12}
$$

We see from the latter that the circle is converted to an oblique ellipse via two operations. The first is an operation of scaling which produces an ellipse whose principal axes are aligned with those of the coordinate system. The second operation is a rotation which tilts the ellipse.

The vectors of the matrix that effects the rotation define the axes of the ellipse. They have the property that, when they are mapped through the matrix A, their orientation is preserved and only their length is altered. Thus

$$
\begin{aligned}
&\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}.
\end{aligned} \tag{13}
$$

Such vectors are described as the characteristic vectors of the matrix, and the factors λ_1 and λ_2, by which their lengths are altered under the transformation, are described as the corresponding characteristic roots.