MIXED ESTIMATION

The procedure which is described as mixed estimation is sometimes the appropriate way of combining information relating to the parameters of a regression equation which comes from two or more sources or which comes at different periods from the same source. The technique is, in fact, the classical counterpart of a Bayesian technique which can be expressed in terms of the calculus of conditional expectations. It also has some affinity with the method of restricted least-square estimation which provides the appropriate means of estimating a regression equation in which the parameter vector is know to fulfil a set of exactly specified constraints.

To describe the method of mixed estimation, let us imagine that there is some information about the parameter vector β which can be expressed in a set of equations of the form

$$(1) \qquad r = R\beta + \upsilon, \quad \text{where} \quad E(\upsilon) = 0 \quad \text{and} \quad D(\upsilon) = E(\upsilon\upsilon') = V.$$

For example, there might be some doubt concerning the validity of the restrictions which set $R\beta = r$. The doubt can be expressed by supplementing the equations of the restrictions by a random vector υ. In that case, the diagonal elements of the matrix V, which is the dispersion matrix of υ, can be regarded as measures of the uncertainty affecting the individual constraints. The larger is the value of an element, the greater is the uncertainty of the information and the less should be the reliance placed upon it.

Imagine that there is also a set of regular observations on the regression model which can be expressed in the equations

$$(2) \qquad y = X\beta + \varepsilon, \quad \text{where} \quad E(\varepsilon) = 0 \quad \text{and} \quad D(\varepsilon) = E(\varepsilon\varepsilon') = \sigma^2 Q_T.$$

The two sets of equations under (1) and (2) can now be joined to form the following system:

$$(3) \qquad \begin{bmatrix} y \\ r \end{bmatrix} = \begin{bmatrix} X \\ R \end{bmatrix} + \begin{bmatrix} \varepsilon \\ \upsilon \end{bmatrix},$$

wherein

$$(4) \qquad E\begin{bmatrix} \varepsilon \\ \upsilon \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad D\begin{bmatrix} \varepsilon \\ \upsilon \end{bmatrix} = \begin{bmatrix} \sigma^2 Q & 0 \\ 0 & V \end{bmatrix}.$$

Here the dispersion matrix is constructed on the assumption that the covariance matrix of ε and υ is $C(\varepsilon, \upsilon) = 0$.

Let us define some matrices and vectors which combine the elements of equations (1) and (2):

$$(5) \qquad q = \begin{bmatrix} y \\ r \end{bmatrix}, \qquad W = \begin{bmatrix} X \\ R \end{bmatrix}, \qquad \eta = \begin{bmatrix} \varepsilon \\ \upsilon \end{bmatrix}.$$

Then the combined system can be written in the form of

$$q = W\beta + \eta, \quad \text{where} \quad E(\eta) = 0 \quad \text{and} \quad D(\eta) = E(\eta\eta') = \sigma^2\Omega. \tag{6}$$

The efficient generalised least-squares estimator of the parameter vector β is now

$$\hat{\beta} = (W'\Omega^{-1}W)^{-1}W'\Omega^{-1}y. \tag{7}$$

When the explicit forms of W and q are put in place, this expression becomes

$$\hat{\beta} = \Big(\frac{1}{\sigma^2}X'Q^{-1}X + R'V^{-1}R\Big)^{-1}\Big(\frac{1}{\sigma^2}X'Q^{-1}y + R'V^{-1}r\Big). \tag{8}$$

The matrix

$$D(\hat{\beta}) = \Big(\frac{1}{\sigma^2}X'Q^{-1}X + R'V^{-1}R\Big)^{-1} \tag{9}$$

is the dispersion matrix of the estimates.

It is worth remarking here that the necessary and sufficient condition for the existence of a set of uniquely defined generalised least-squares estimates is that this inverse matrix must exist. We may assume that Q and V are positive definite matrices, which means that they are invertible. It follows that the inverse matrix of (9) exists if and only if the combined matrix W has full column rank. This represents the requirement that, taken together, the two sources will provide enough information to allow the estimates to be formed.

Either source of information, taken on its own, may be insufficient for the purpose. Indeed, the very fact that two separate information sources are being tapped may be an indication that neither on them, on its own, is sufficient.