THE GEOMETRIC LAG SCHEME

Anr early approach to the problem of defining a lag structure which depends on a limited number of parameters was that of Koyk who proposed the following geometric lag scheme:

$$y(t) = \beta\big\{x(t) + \phi x(t-1) + \phi^2 x(t-2) + \cdots\big\} + \varepsilon(t). \tag{1}$$

Here, although we have an infinite set of lagged values of $x(t)$, we have only two parameters which are β and ϕ.

It can be seen that the impulse-response function of the Koyk model takes a very restricted form. It begins with an immediate response to the impulse. Thereafter, the response dies away in the manner of a convergent geometric series, or of a decaying exponential function of the sort which also characterises processes of radioactive decay.

The values of the coefficients in the Koyk distributed-lag scheme tend asymptotically to zero; and so it can said that the full response is never accomplished in a finite time. To characterise the speed of response, we may calculate the median lag which is analogous to the half-life of a process of radioactive decay. The gain of the transfer function, which is obtained by summing the geometric series $\{\beta, \phi\beta, \phi^2\beta, \ldots\}$, has the value of

$$\gamma = \frac{\beta}{1-\phi}. \tag{2}$$

To make the Koyk model amenable to estimation, we might first transform the equation. By lagging the equation by one period and multiplying the result by ϕ, we get

$$\phi y(t-1) = \beta\big\{\phi x(t-1) + \phi^2 x(t-2) + \phi^3 x(t-3) + \cdots\big\} + \phi\varepsilon(t-1). \tag{3}$$

Taking the latter from (1) gives

$$y(t) - \phi y(t-1) = \beta x(t) + \big\{\varepsilon(t) - \phi\varepsilon(t-1)\big\}. \tag{4}$$

With the use of the lag operator, we can write this as

$$(1-\phi L)y(t) = \beta x(t) + (1-\phi L)\varepsilon(t), \tag{5}$$

of which the rational form is

$$y(t) = \frac{\beta}{1-\phi L}x(t) + \varepsilon(t). \tag{6}$$

In fact, by using the expansion

$$\begin{aligned}\frac{\beta}{1-\phi L}x(t) &= \beta\big\{1 + \phi L + \phi L^2 + \cdots\big\}x(t)\\ &= \beta\big\{x(t) + \phi x(t-1) + \phi x(t-2) + \cdots\big\}\end{aligned} \tag{7}$$

within equation (6), we can recover the original form under (1).

Equation (4) is not amenable to consistent estimation by ordinary least squares regression. The reason is that the composite disturbance term $\{\varepsilon(t)-\phi\varepsilon(t-1)\}$ is correlated with the lagged dependent variable $y(t-1)$—since the elements of $\varepsilon(t-1)$ form part of the contemporaneous elements of $y(t-1)$. This conflicts with one of the basic conditions for the consistency of ordinary least-squares estimation which is that the disturbances must be uncorrelated with the regressors. Nevertheless, there is available a wide variety of simple procedures for estimating the parameters of the Koyk model consistently.

One of the simplest procedures for estimating the geometric-lag scheme is based on the original form of the equation under (1). In view of that equation, we may express the elements of $y(t)$ which fall within the sample as

$$\begin{aligned} y_t &= \beta\sum_{i=0}^{\infty}\phi^i x_{t-i}+\varepsilon_t \\ &= \theta\phi^t+\beta\sum_{i=0}^{t-1}\phi^i x_{t-i}+\varepsilon_t \\ &= \theta\phi^t+\beta z_t+\varepsilon_t. \end{aligned} \tag{8}$$

Here

$$\theta=\beta\big\{x_0+\phi x_{-1}+\phi^2 x_{-2}+\cdots\big\} \tag{9}$$

is a nuisance parameter which embodies the presample elements of the sequence $x(t)$, whilst

$$z_t=x_t+\phi x_{t-1}+\cdots+\phi^{t-1}x_1 \tag{10}$$

is an explanatory variable compounded from the observations $x_t, x_{t-1},\ldots,x_1$ and from the value attributed to ϕ.

The procedure for estimating ϕ and β which is based on equation (8) involves running a number of trial regressions with differing values of ϕ and therefore of the regressors ϕ^t and z_t; $t=1,\ldots,T$. The definitive estimates are those which correspond to the least value of the residual sum of squares.

It is possible to elaborate this procedure so as to obtain the estimates of the parameters of the equation

$$y(t)=\frac{\beta}{1-\phi L}x(t)+\frac{1}{1-\rho L}\varepsilon(t), \tag{11}$$

which has a first-order autoregressive disturbance scheme in place of the white-noise disturbance to be found in equation (6). An estimation procedure may be devised which entails searching for the optimal values of ϕ and ρ within the square defined by $-1<\rho,\phi<1$. There may even be good reason to suspect that these values will be found within the quadrant defined by $0\leq\rho,\phi<1$.

The task of finding estimates of ϕ and ρ is assisted by the fact that we can afford, at first, to ignore autoregressive nature of the disturbance process while searching for the optimum value of the systematic parameter ϕ.

When a value has been found for ϕ, we shall have residuals which are consistent estimates of the corresponding disturbances. Therefore, we can proceed to fit the AR(1) model to the residuals in the knowledge that we will then be generating a consistent estimate of the parameter ρ; and, indeed, we can might use ordinary least-squares regression for this purpose. Having found the estimate for ρ, we should wish to revise our estimate of ϕ.

Lagged Dependent Variables

In spite of the relative ease with which one may estimate the Koyk model, it has been common throughout the history of econometrics to adopt an even simpler approach in the attempt to model the systematic dynamics.

Perhaps the easiest way of setting a regression equation in motion is to include a lagged value of the dependent variable on the RHS in the company of the explanatory variable x. The resulting equation has the form of

$$(12) \qquad y(t) = \phi y(t-1) + \beta x(t) + \varepsilon(t).$$

In terms of the lag operator, this is

$$(13) \qquad (1-\phi L)y(t) = \beta x(t) + \varepsilon(t),$$

of which the rational form is

$$(14) \qquad y(t) = \frac{\beta}{1-\phi L}x(t) + \frac{1}{1-\phi L}\varepsilon(t).$$

The advantage of equation (12) is that it is amenable to estimation by ordinary least-squares regression. Although the estimates will be biased in finite samples, they are, nevertheless, consistent in the sense that they will tend to converge upon the true values as the sample size increases—provided, of course, that the model corresponds to the processes underlying the data.

The model with a lagged dependent variable generates precisely the same geometric distributed-lag schemes as does the Koyk model. This can be confirmed by applying the expansion given under (7) to the rational form of the present model given in equation (14) and by comparing the result with (1). The comparison of equation (14) with the corresponding rational equation (6) for the Koyk model shows that we now have an AR(1) disturbance process described by the equation

$$(15) \qquad \eta(t) = \phi\eta(t-1) + \varepsilon(t)$$

in place of a white-noise disturbance $\varepsilon(t)$.

This might be viewed as an enhancement of the model were it not for the constraint that the parameter ϕ in the systematic transfer function is the same as the parameter ϕ

in the disturbance transfer function. For such a constraint is appropriate only if it can be argued that the disturbance dynamics are the same as the systematic dynamics—and they need not be.

To understand the detriment of imposing the constraint, let us imagine that the true model is of the form given under (11) with ρ and ϕ taking very different values. Imagine that, nevertheless, it is decided to fit the equation under (14). Then the estimate of ϕ will be a biased and an inconsistent one whose value falls somewhere between the true values of ρ and ϕ in equation (11). If this estimate of ϕ is taken to represent the systematic dynamics of the model, then our inferences about such matters as the speed of convergence of the impulse response and the value of the steady-state gain are liable to be misleading.

Partial Adjustment and Adaptive Expectations

There are some tenuous justifications both for the Koyk model and for the model with a lagged dependent variable which arise from economic theory.

Consider a partial-adjustment model of the form

$$y(t) = \lambda\big\{\gamma x(t)\big\} + (1-\lambda)y(t-1) + \varepsilon(t), \tag{16}$$

where, for the sake of a concrete example, $y(t)$ is current consumption, $x(t)$ is disposable income and $\gamma x(t) = y^*(t)$ is "desired"consumption. Here we are supposing that habits of consumption persist, so that what is consumed in the current period is a weighted combination of the previous consumption and present desired consumption. The weights of the combination depend on the partial-adjustment parameter $\lambda \in (0,1]$. If $\lambda = 1$, then the consumers adjust their consumption instantaneously to the desired value. As $\lambda \to 0$, their consumption habits become increasingly persistent. When the notation $\lambda\gamma = \beta$ and $(1-\lambda) = \phi$ is adopted, equation (16) becomes identical to equation (12) which relates to a simple regression model with a lagged dependent variable.

An alternative model of consumers' behaviour derives from Friedman's Permanent Income Hypothesis. In this case, the consumption function is specified as

$$y(t) = \delta x^*(t) + \varepsilon(t), \tag{17}$$

where

$$\begin{aligned} x^*(t) &= (1-\phi)\big\{x(t) + \phi x(t-1) + \phi^2 x(t-2) + \cdots\big\} \\ &= \frac{1-\phi}{1-\phi L}x(t) \end{aligned} \tag{18}$$

is the value of permanent or expected income which is formed as a geometrically weighted sum of all past values of income. Here it is asserted that a consumer plans his expenditures in view of his customary income, which he assesses by taking a long view over all of his past income receipts.

An alternative expression for the sequence of permanent income is obtained by multiplying both sides of (18) by $1 - \phi L$ and rearranging the result. Thus

$$x^*(t) - x^*(t-1) = (1-\phi)\big\{x(t) - x^*(t-1)\big\}, \tag{19}$$

which depicts the change of permanent income as a fraction of the prediction error $x(t) - x^*(t-1)$. The equation depicts a so-called adaptive-expectations mechanism.

On substituting the expression for permanent income under (18) into the equation (17) of the consumption function, we get

$$y(t) = \delta \frac{(1-\phi)}{1-\phi L} x(t) + \varepsilon(t). \tag{20}$$

When the notation $\delta(1-\phi) = \beta$ is adopted, equation (20) becomes identical to the equation (6) of the Koyk model.