WEAK EXOGENEITY

With the algebra of the bivariate Normal distribution at our disposal, we are in a position to discuss the concept of exogeneity. As its etymology suggests, this term refers to variables that are generated outside the system or the equation of interest. The dependent variables, that are generated within the system, are described as endogenous.

In so far as their values affect those of the dependent variables, the exogenous variables are apt to be described as explanatory variables or regressors. The concept of exogeneity is appropriate to circumstances where regression equations correspond to components of the economy that can be regarded as structural entities embodying causal relationships running from the explanatory variables to the dependent variables.

The parameters of a structural regression equation are its intrinsic properties; and they are expected to be invariant in respect of any changes in the circumstances affecting the generation of the exogenous variables. Moreover, the validity of the ordinary methods of regression analysis are dependent upon the truth of the assumption that the disturbance term is uncorrelated with the explanatory variables that are regarded as exogenous.

There are also circumstances where regression equations represent a statistical relationships that corresponds neither to a structural relationship nor to a causal connection. The parameters of such a regression equation are a reflection of the joint distribution of the variables comprised by the equation. They are expected to remain constant only in so far as the joint distribution is unchanged.

In the case of a purely statistical regression equation, it is generally inappropriate to categorise the variables as exogenous or endogenous. Moreover, the role of the disturbance term of the structural regression equation, which is deemed to represent the aggregate effect of the omitted exogenous variables, is taken by the prediction error, which, by construction, is uncorrelated with the regressors.

The concept of exogeneity has been analysed and elaborated in an influential article of Engle, Hendry and Richard (1983). Their analysis is concerned primarily with structural regression equations. Nevertheless, it goes some way towards bridging the gap that exists between the structural and the statistical interpretations. It must be said that, in the process, the authors have altered the meaning of the word exogeneity to the extent that they are prepared to find conditions of weak exogeneity in equations that are devoid of any structural or behavioural interpretation.

The discussion of exogeneity tends to be heavily burdened by special definitions and by neologisms, but we shall attempt to convey the basic ideas in the simple of terms and within the context of bivariate relationships.

We should begin by noting that there is nothing inherent is the structure of a bivariate distribution to indicate which of the variables is the dependent variable and which is the explanatory variable. The two variables are appointed to play these roles by choosing one or other of the available factorisations that depict the joint distribution as the product of a marginal distribution and a conditional distribution:

$$f(x,y)=f(y|x)f(x)=f(x|y)f(y). \tag{1}$$

We shall choose y as the dependent variable and x as the regressor. In that case, it is the conditional distribution $f(y|x)$ that embodies the regression equation. If x qualifies as a (weakly) exogenous variable, then we are in a position to ignore the details of the marginal distribution $f(x)$ when making inferences about the parameters of the conditional distribution.

In terms of the existing notation, we have

$$\begin{aligned}E(y|x)&=\mu_{y|x}=\alpha+\beta x,\\ E(x)&=\mu_x,\end{aligned} \tag{2}$$

where

$$\beta=\frac{\sigma_{xy}}{\sigma_{xy}}=\frac{\rho\sigma_y}{\sigma_x}\qquad\text{and}\qquad\alpha=\mu_y-\beta\mu_x. \tag{3}$$

We may also define the disturbance terms of the conditional and the marginal distributions, which are

$$y_t-\mu_{y|x}=\varepsilon_t\sim N(0,\sigma^2)\qquad\text{and}\qquad x_t-\mu_x=\nu_t\sim N(0,\sigma_x^2). \tag{4}$$

By construction, these are statistically independent with $C(\varepsilon_t,\nu_t)=0$. There is also a specification for $V(\varepsilon_t)=\sigma^2$:

$$\sigma^2=\sigma_y^2(1-\rho^2)=\sigma_{yy}-\frac{\sigma_{xy}^2}{\sigma_{xx}}. \tag{5}$$

The factorisation of the bivariate distribution has entailed the replacement of the parameter set $\Phi=\{\mu_y,\mu_x,\sigma_x^2,\sigma_y^2,\rho\}$ by two parameter sets, which are $\Lambda_1=\{\alpha,\beta,\sigma^2\}$ and $\Lambda_2=\{\mu_x,\sigma_x^2\}$. To show the dependence of the distributions upon the parameters, we may write the chosen factorisation as

$$f(x_t,y_t;\Phi)=f(y_t|x_t;\Lambda_1)f(x_t;\Lambda_2). \tag{6}$$

We define the parameters of interest to be a function $\Psi=g(\Lambda_1)$ of the parameters of the conditional distribution.

We are now in a position to supply the central definition of Engle *et al.*:

(7) The variable x_t is said to be *weakly exogenous* for $\Psi=g(\Lambda_1)$ if and only if the factorisation of (6) generates a parameter space $\Lambda=\Lambda_1\times\Lambda_2=\{(\lambda_1,\lambda_2)\}$ in which the elements $\lambda_1\in\Lambda_1$ and $\lambda_2\in\Lambda_2$ are free to vary independently of each other.

This definition is concerned essentially with the efficiency of estimation. Thus, a variable x_t is defined to be weakly exogenous for the purposes of estimating the parameters of interest if it entails no loss of information to confine ones attention to the conditional distribution of y_t given x_t and to disregard the marginal distribution of x_t.

The definition would normally allow us to describe the argument x_t of the marginal distribution $f(x_t)$, as well as the argument y_t of $f(y_t)$, as an exogenous variable when there is no structural information regarding the parameters of Λ_1 and Λ_2 to constrain their independent variation. However, the full implications of the definition are best understood in the context of a simple example.

Example. Consider the so-called cobweb model that depicts an agricultural market in which the price is determined in consequence of the current supply and in which the supply has been determined by the price of the previous period. The resulting structural equations are

$$p_t = \beta q_t + \varepsilon_t, \qquad \text{where} \qquad \varepsilon_t \sim N(0, \sigma^2), \tag{8}$$

$$q_t = \pi_q p_{t-1} + \nu_{qt}, \qquad \text{where} \qquad \nu_{qt} \sim N(0, \sigma_q^2). \tag{9}$$

Here, p_t and q_t are the logarithms of price and quantity respectively, which have been adjusted by subtracting the sample means in order to eliminate the intercept terms from the equations. They are in place of y_t and x_t respectively. The value of $1/\beta$ is the price elasticity of demand, and that of π_q is the price elasticity of supply.

It is assumed that $C(\varepsilon_t, \nu_{qt}) = 0$, which reflects the fact that the circumstances in which the agricultural product is created are remote from those in which it is marketed. It follows that $C(q_t, \nu_{qt}) = 0$, which guarantees that q_t is exogenous with respect to equation (8) in the conventional sense. However, in view of the feedback that runs from (8) to (9), we choose to describe q_t as a predetermined variable rather than an exogenous variable.

The variables p_t and q_t also have a purely statistical joint distribution with constant mean values, which gives rise to what are described as the reduced-form equations:

$$p_t = \pi_p p_{t-1} + \nu_{pt}, \qquad \text{where} \qquad \nu_{pt} \sim N(0, \sigma_p^2), \tag{10}$$

$$q_t = \pi_q p_{t-1} + \nu_{qt}, \qquad \text{where} \qquad \nu_{qt} \sim N(0, \sigma_q^2). \tag{11}$$

It is assumed that $C(\nu_{pt}, \nu_{qt}) = \sigma_{pq} \neq 0$.

From the conditional distribution of p_t given p_t, we obtain

$$\begin{aligned} E(p_t|q_t) &= E(p_t) + \frac{C(p_t, q_t)}{V(q_t)}\{q_t - E(q_t)\} \\ &= \pi_p p_{t-1} + \frac{\sigma_{pq}}{\sigma_{qq}}\{q_t - \pi_q p_{t-1}\}, \end{aligned} \tag{12}$$

from which

$$p_t = (\pi_p - \beta\pi_q)p_{t-1} + \beta q_t + \varepsilon_t. \tag{13}$$

Here, ε_t is, by construction, uncorrelated with ν_{qt}.

The comparison of equation (13) with equation (8) makes it clear that the cobweb model embodies the restriction that $\pi_p - \beta\pi_q = 0$. (Notice that, in the absence of the condition $C(\varepsilon_t, \nu_{qt}) = 0$ affecting equations (8) and (9), equation (13) could not be identified with (8).)

Imagine that $\beta \in \Lambda_1$ alone is the parameter of interest. Then the restriction that excludes p_{t-1} from equation (8) will make q_t weakly exogenous for β.

There is, however, the matter of the dynamic stability of the system to be considered. It is natural to assume that the disturbances to the cobweb system will give rise to damped cycles in the prices and quantities. This necessitates that the coefficient of equation (10) obeys the condition that $|\pi_p| < 1$. Substitution of equation (9) into equation (8) gives

$$p_t = \beta\pi_q p_{t-1} + (\varepsilon_t + \beta\nu_{qt}). \tag{14}$$

This is an alternative rendering of equation (10) which shows again that $\pi_p = \beta\pi_q$.

When the stability of the system is taken into account, there is a connection between the parameters $\beta \in \Lambda_1$ and $\pi_q \in \Lambda_2$ of the conditional and the marginal distributions, such that their values are inversely related. Knowing the value of π_q will enable one to delimit the permissible values of β. Thus, in circumstances where the dynamic stability of the system is a necessary assumption, the variable x_t is no longer weakly exogenous for β according to the definition of (7).

The foregoing example can be used to illustrate two further definitions that have been cited by Engle *et al.*:

(15) The variable q_t is said to be *predetermined* in equation (8) if and only if $C(q_t, \varepsilon_{t+i}) = 0$ for all $i \geq 0$, whereas it is said to be *strictly exogenous* in (8) if and only if $C(q_t, \varepsilon_{t+i}) = 0$ for all i.

These are, in fact, the conventional definitions of exogenous and predetermined variables. It is notable that they make reference only to the equation in question and not to the parameters of interest therein.

For the condition of strict exogeneity to be satisfied in equation (8), it would be necessary to eliminate the feedback to equation (9). This would suggest that the producers have no regard to previous or current prices.

Reference

Engle, R.F., D.F. Hendry and J.-F. Richard, (1983), Exogeneity, *Econometrica,* 51, 277–304.