ERROR CORRECTION, NONSTATIONARITY AND COINTERGRATED VARIABLES

In this note, we shall consider some of the methods that are available for modelling tranfer function relationships which map non-stationary input sequences into nonstationary outputs.

We can begin by considering a first-order simple dynamic model of the form

$$y(t) = \phi y(t-1) + x(t)\beta + \varepsilon(t). \tag{1}$$

Taking $y(t-1)$ from both sides of this equation gives

$$\begin{aligned}\nabla y(t) = y(t) - y(t-1) &= (\phi - 1)y(t-1) + \beta x(t) + \varepsilon(t)\\ &= (1-\phi)\left\{\frac{\beta}{1-\phi}x(t) - y(t-1)\right\} + \varepsilon(t)\\ &= \lambda\big\{\gamma x(t) - y(t-1)\big\} + \varepsilon(t),\end{aligned} \tag{2}$$

where $\lambda = 1-\phi$ and where γ is the gain of the transfer function which maps from $x(t)$ to $y(t)$. This is the so-called error-correction form of the equation; and it indicates that the change in $y(t)$ is a function of the extent to which the proportions of the series $x(t)$ and $y(t-1)$ differs from those which would prevail in the steady state.

The error-correction form provides the basis for estimating the parameters of the model when the signal series $x(t)$ is trended or nonstationary. In such circumstances, it is easy to obtain an accurate estimate of γ simply by running a regression of $y(t-1)$ on $x(t)$; for all that is required of the regression is that it should determine the fundamental coefficient of proportionality which, in the long term, dominates the relationship which exists between the two series. Once a value for γ is available, the remaining parameter λ may be estimated by regressing $\nabla y(t)$ upon the composite variable $\{\gamma x(t) - y(t-1)\}$.

It is possible to derive an error-correction form for a more general model denoted by

$$y(t) = \phi_1 y(t-1) + \cdots + \phi_p y(t-p) + \beta_0 x(t) + \cdots + \beta_k x(t-k) + \varepsilon(t). \tag{3}$$

We can proceed to reparametrise this model so that it assumes the equivalent form of

$$\begin{aligned}y(t) = \theta y(t-1) &+ \rho_1 \nabla y(t-1) + \cdots + \rho_p \nabla y(t-p+1)\\ &+\kappa x(t) + \delta_0 \nabla x(t) + \cdots + \delta_k \nabla x(t-k+1) + \varepsilon(t),\end{aligned} \tag{4}$$

where $\theta = \phi_1 + \cdots + \phi_p$ and $\kappa = \beta_0 + \cdots + \beta_k$. Now let us subtract $y(t-1)$ from both sides of equation (4). This gives

$$\begin{aligned}\nabla y(t) =&(\theta - 1)y(t-1) + \kappa x(t)\\ &+ \rho_1 \nabla y(t-1) + \cdots + \rho_p \nabla y(t-p+1)\\ &+ \delta_0 \nabla x(t) + \cdots + \delta_k \nabla x(t-k+1) + \varepsilon(t).\end{aligned} \tag{5}$$

The first two terms on the RHS combine to give

$$
\begin{aligned}
(\theta - 1)y(t-1) + \kappa x(t) &= (1-\theta)\left\{ \frac{\kappa}{1-\theta} x(t) - y(t-1) \right\} \\
&= \lambda\big\{\gamma x(t) - y(t-1)\big\}
\end{aligned} \tag{6}
$$

which is an error-correction term in which γ is the value of the gain of the tranfer function. It follows that the error-correction form of equation (3) is

$$
\nabla y(t) = \lambda\big\{\gamma x(t) - y(t-1)\big\} + \sum_{i=1}^{p-1} \rho_i \nabla y(t-i) + \sum_{i=0}^{k-1} \delta_i \nabla x(t-i) + \varepsilon(t). \tag{7}
$$

In the case of a nonstationary signal $x(t)$, this is amenable to precisely the same principle of estimation as was the simpler first-order equation equation under (2). That is to say, we can begin by estimating the gain γ by a simple regression of $y(t-1)$ on $x(t)$. Then, when a value for γ is available, we can proceed to find the remaining parameters of the model via a second regression.

Example. To reveal the nature of the reparameterisation which transforms equation (3) into equation (4), let us consider the following example:

$$
\begin{aligned}
&\beta_0 x(t) + \beta_1 x(t-1) + \beta_2 x(t-2) + \beta_3 x(t-3) \\
&\quad = \big\{\beta_0 + \beta_1 + \beta_2 + \beta_3\big\}x(t) - \big\{\beta_1 + \beta_2 + \beta_3\big\}\big\{x(t) - x(t-1)\big\} \\
&\qquad\qquad - \big\{\beta_2 + \beta_3\big\}\big\{x(t-1) - x(t-2)\big\} \\
&\qquad\qquad - \beta_3\big\{x(t-2) - x(t-3)\big\} \\
&\quad = \kappa x(t) + \delta_0 \nabla x(t) + \delta_1 \nabla x(t-1) + \delta_2 \nabla x(t-2).
\end{aligned} \tag{8}
$$

The example may be systematised. Consider the product $\beta' x$ wherein $x = [x(t), x(t-1), x(t-2), x(t-3)]'$ and $\beta' = [\beta_0, \beta_1, \beta_2, \beta_3]$. Let Λ be an arbitrary nonsingular, i.e. invertible, matrix of order 4×4. Then $\beta' x = \{\beta' \Lambda^{-1}\}\{\Lambda x\} = \delta' z$ where $z = \Lambda x$ and $\delta' = \beta' \Lambda^{-1}$. That is to say, the expression in terms of z and δ is equivalent to the original expression in terms of x and β. With these results in mind, let us consider the following transformations:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\begin{bmatrix} x(t) \\ x(t-1) \\ x(t-2) \\ x(t-3) \end{bmatrix}
=
\begin{bmatrix} x(t) \\ -\nabla x(t) \\ -\nabla x(t-1) \\ -\nabla x(t-2) \end{bmatrix} \tag{9}
$$

and

$$
\begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}
= \begin{bmatrix} \kappa & -\delta_0 & -\delta_1 & -\delta_2 \end{bmatrix}. \tag{10}
$$

Here the two matrices which affect the transformation upon the variables and upon their associated parameters stand in an inverse relationship to one another. They are, in fact, the matrix analogues, repsctively, of the operators $1-L$ and $(1-L)^{-1} = \{1 + L + L^2 + \cdots\}$. The transformations provide a simple example of what is entailed in converting equation (3) into equation (4).