COINTEGRATED BIVARIATE VECTOR AUTOREGRESSION

Consider a bivariate vector autoregressive system of the first order driven by normally distributed disturbances that form cross-correlated white-noise sequences. This can be written in the form of

$$(1)\qquad \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \nu_{1t} \\ \nu_{2t} \end{bmatrix},$$

where

$$(2)\qquad E\begin{bmatrix} \nu_{1t} \\ \nu_{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad D\begin{bmatrix} \nu_{1t} \\ \nu_{2t} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix}.$$

Our objective is to reparametrise these equations to produce a conditional and a marginal autoregressive distributed-lag model. Thereafter, we shall cast the conditional equation in an error-correction form.

Let $\mathcal{I}_{t-1}$ denote the information available at time $t-1$. We may observe that, given the Markov structure of the system, the sample information is summarised completely by the lagged values y_{t-1}, z_{t-1}. Therefore, $\mathcal{I}_{t-1}$ contains only these two sample values together with the parameters of the system.

The objective is to find the equations that provide the values of the marginal expectation $E(z_t|\mathcal{I}_{t-1})$ and of the conditional expectation

$$(3)\quad E(y_t|z_t,\mathcal{I}_{t-1}) = E(y_t|\mathcal{I}_{t-1}) + C(y_t,z_t|\mathcal{I}_{t-1})D^{-1}(z_t|\mathcal{I}_{t-1})\{z_t - E(z_t|\mathcal{I}_{t-1})\}.$$

We already have

$$(4)\qquad E(y_t|\mathcal{I}_{t-1}) = \pi_{11}y_{t-1} + \pi_{12}z_{t-1},$$

$$(5)\qquad E(z_t|\mathcal{I}_{t-1}) = \pi_{21}y_{t-1} + \pi_{22}z_{t-1}.$$

There are also

$$(6)\qquad \begin{aligned} &C(y_t,z_t|\mathcal{I}_{t-1}) = \omega_{12}, \qquad D(z_t|\mathcal{I}_{t-1}) = \omega_{22},\\ &z_t - E(z_t|\mathcal{I}_{t-1}) = z_t - \pi_{21}y_{t-1} + \pi_{22}z_{t-1}. \end{aligned}$$

Therefore, on substituting (4)–(6) into (3), it transpires that

$$(7)\qquad E(y_t|z_t,\mathcal{I}_{t-1}) = \pi_{11}y_{t-1} + \pi_{12}z_{t-1} + \frac{\omega_{12}}{\omega_{22}}\{z_t - \pi_{21}y_{t-1} - \pi_{22}z_{t-1}\}.$$

It follows that, by adding the disturbance terms to the two expectations of (7) and (5) to give $y_t = E(y_t|z_t,\mathcal{I}_{t-1}) + \eta_t$ and $z_t = E(z_t|\mathcal{I}_{t-1}) + \nu_{2t}$, we get

$$(8)\qquad y_t = \beta_0 z_t + \beta_1 z_{t-1} + \beta_2 y_{t-1} + \eta_t, \qquad \eta_t \sim N(0,\sigma^2),$$

$$(9)\qquad z_t = \pi_{22}z_{t-1} + \pi_{21}y_{t-1} + \nu_{2t}, \qquad \nu_{2t} \sim N(0,\omega_{22}),$$

where

$$\beta_0 = \frac{\omega_{12}}{\omega_{22}}, \tag{10}$$

$$\beta_1 = \pi_{12} - \frac{\omega_{12}}{\omega_{22}}\pi_{22}, \tag{11}$$

$$\beta_2 = \pi_{11} - \frac{\omega_{12}}{\omega_{22}}\pi_{21}. \tag{12}$$

It is straightforward to derive an error-correction version of equation (8). Thus, taking y_{t-1} from both sides of the equation and rearranging the RHS gives

$$\nabla y_t = \beta_0 \nabla z_t + \lambda_1 (y_{t-1} - \delta z_{t-1}) + \eta_t, \tag{13}$$

where

$$\lambda_1 = \beta_2 - 1, \tag{14}$$

$$\delta = \frac{-(\beta_0 + \beta_1)}{\beta_2 - 1}. \tag{15}$$

Integrated Variables

Now let us imagine the variables y_t and z_t of equation (1) follow random walks. By taking the vector $[y_{t-1}, z_{t-1}]'$ from both sides of the equation, we get

$$\begin{bmatrix} \nabla y_t \\ \nabla z_t \end{bmatrix} = \begin{bmatrix} \pi_{11} - 1 & \pi_{12} \\ \pi_{21} & \pi_{22} - 1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \nu_{1t} \\ \nu_{2t} \end{bmatrix}. \tag{16}$$

Here the differenced variables ∇y_t, ∇z_t represent stationary stochastic processes, whereas y_t and z_t are nonstationary. The two sides of the equation (16) can be reconciled only if the matrix transformation of $[y_{t-1}, z_{t-1}]'$ on the RHS results in a vector of stationary variables.

This will be impossible if the matrix is of full rank. It will be possible to reconcile the two sides of (16) if the matrix is of zero rank—which is to say that it has zero-valued elements—or else if it has a rank of unity. In the latter case, the matrix must be expressible as the outer product of a column vector $[\alpha_1, \alpha_2]'$ and a row vector $[1, -\delta]$, of which the leading element may be normalised without loss of generality. In effect, equation (1) becomes

$$\begin{bmatrix} \nabla y_t \\ \nabla z_t \end{bmatrix} = \begin{bmatrix} \alpha_1 & -\alpha_1\delta \\ \alpha_2 & -\alpha_2\delta \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \nu_{1t} \\ \nu_{2t} \end{bmatrix}, \tag{17}$$

wherein $y_{t-1} - \delta z_{t-1}$ is a stationary combination on account of a long-run proportionality that is maintained by y_{t-1} and z_{t-1}, which follow a common stochastic trend.

Notwithstanding the non-stationarity of the variables y_t, z_t, it remains possible to factorise their joint distribution as the product of a marginal distribution and a conditional distribution. Also, the error-correction formulation of (13) remains valid. However, it now becomes appropriate to express equation (9) in terms of differenced and cointegrated variables. Thus, by taking z_{t-1} from both sides, we get

$$\nabla z_t = (\pi_{22} - 1)z_{t-1} + \pi_{21}y_{t-1} + \nu_{2t}. \tag{18}$$

But $\pi_{22} - 1 = -\alpha_2\delta$ and $\pi_{21} = \alpha_2$, so the conditional and the marginal equations can be written together as

$$\nabla y_t = \beta_0\nabla z_t + \lambda_1(y_{t-1} - \delta z_{t-1}) + \eta_t, \tag{19}$$

$$\nabla z_t = \alpha_2(y_{t-1} - \delta z_{t-1}) + \nu_{2t}, \tag{20}$$

where

$$\beta_0 = \frac{\omega_{12}}{\omega_{22}}, \tag{21}$$

$$\begin{aligned}\lambda_1 = \beta_2 - 1 &= \pi_{11} - \frac{\omega_{12}}{\omega_{22}}\pi_{12} - 1\\ &= \alpha_1 - \frac{\omega_{12}}{\omega_{22}}\alpha_2.\end{aligned} \tag{22}$$

Observe that $C(\eta_t, \nu_{2t}) = 0$, by construction, and that we have $C(\eta_t, \nabla z_t) = 0$ in consequence. The latter condition reflects nothing more than the fact that we are using ∇z_t as a regressor; and, in these circumstances, (19) does not, in general, represent a structural equation.

The structural version of equation (19), can be accompanied by a structural equation for z_t to form the system

$$\nabla y_t = \gamma_{11}\nabla z_t + \gamma_{12}(y_{t-1} - \delta z_{t-1}) + \varepsilon_{1t}, \tag{23}$$

$$\nabla z_t = \gamma_{21}\nabla y_t + \gamma_{22}(y_{t-1} - \delta z_{t-1}) + \varepsilon_{2t}, \tag{24}$$

where

$$E\begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{and} \quad D\begin{bmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\end{bmatrix} = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}. \tag{25}$$

In the absence of further parametric restrictions and without further exogenous variables appearing in either equation, the two structural equations will be unidentifiable.

However, it is possible that the conditions will prevail that will enable us to identify the structural equations (23), (24) with the conditional and marginal equations (19) and (20). The requisite restrictions are that $\gamma_{21} = 0$ and that $\sigma_{12} = \sigma_{12} = 0$, which imply that z_t is predetermined with respect to equation (23). In fact, given that that $\gamma_{21} = 0$, it should be possible directly to assess the validity of the restriction on σ_{12} by examining the value of its empirical counterpart obtained on the supposition that both restrictions are valid.